Q1

Question

The equations of planes, P_1, P_2 are

$$P_1: \quad r = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \beta, \lambda \in \mathrm{R} \qquad P_2 \quad r \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1$$

Find the coordinates of the foot of perpendicular from the point $A(-3, 10, 3)$ to the plane P_2 and show that the point $B(2, 10, -2)$ is the reflection of point A in P_2.

The planes P_1 and P_2 meet in a line L. Find a vector equation in line L.

Plane P_3 is the reflection of P_1 in P_2. Using the results above, find a vector perpendicular to P_3. hence, find, in scalar form, the equation of P_3.

Ans: $F, \left(-\frac{1}{2}, 10, \frac{1}{2}\right), L = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix}, r. \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} = -4$

answer

8 Let F be the foot of perpendicular from A to the plane P_2.

Equation of line AF is given by $r = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \; s \in \mathbb{R}$

Since F lies on the line, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, for some $s \in \mathbb{R}$.

Since F also lies on plane P_2,

$\begin{pmatrix} -3-s \\ 10 \\ 3+s \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1 \quad \Rightarrow (3+s)+(3+s) = 1 \quad \Rightarrow s = -\frac{5}{2}$

Therefore, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} - \frac{5}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 10 \\ \frac{1}{2} \end{pmatrix}$

Coordinates of foot of perpendicular $= \left(-\frac{1}{2}, 10, \frac{1}{2}\right)$.

By ratio theorem, $\overrightarrow{OF} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$

$\Rightarrow \overrightarrow{OB} = 2\overrightarrow{OF} - \overrightarrow{OA} = 2 \begin{pmatrix} -\frac{1}{2} \\ 10 \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix}$

Hence $(2, 10, -2)$ is the reflection of $(-3, 10, 3)$ in P_2.

When the planes intersect,

$\left(\begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1$

$\Rightarrow 6 + 5\lambda = 1$

$\Rightarrow \lambda = -1$

Subst $\lambda = -1$,

$r = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$\therefore$ equation of L is

$r = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; t \in \mathbb{R}$

Alternative:

$\begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix}$

$P_1: \; r \cdot \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix} = -19$

$P_1: 9x + 5y - 14z = -19$

$P_2: -x + z = 1$

By GC, $x = -1 + z$

$y = -2 + z$

$L: r = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; t \in \mathbb{R}$

$A(-3, 10, 3)$ is on P_1, $B(2, 10, -2)$ lies on plane P_3.

Line L also lies on P_3.

Therefore the vector $\begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \\ -2 \end{pmatrix}$ is parallel to P_3.

Hence a vector $\perp$ to P_3 is given by $\begin{pmatrix} 3 \\ 12 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix}$.

Equation of plane P_3 is given by:

$r. \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix} . \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} \Rightarrow r. \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} = -4$

Q2

Question

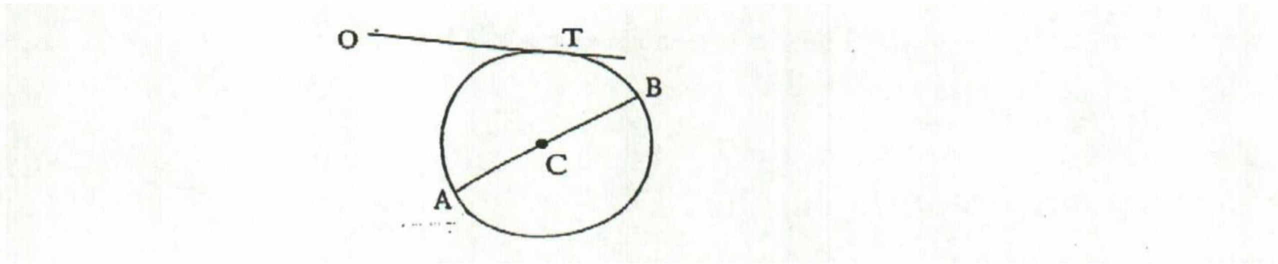
Relative to an origin *O*, the points *A* and *B* have position vectors **a** and **b** respectively,

where $\mathbf{a} \cdot \mathbf{b} > 0$. *A* and *B* also lie on a circle with centre *C* and *AB* as diameter.

(i) Write down the position vector of *C* in terms of **a** and **b**.

(ii) Show that the origin *O* is outside the circle.

T is a point on the circle with position vector **t** and *OT* is a tangent to the circle.



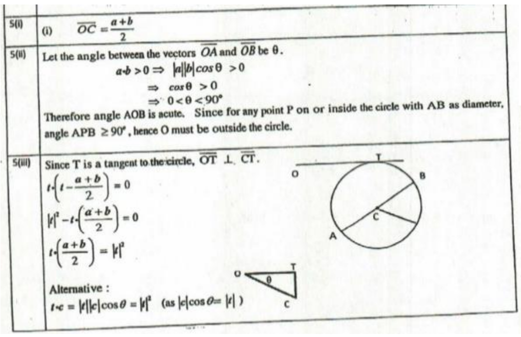
(iii) Show that $\mathbf{t}\cdot\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) = |\mathbf{t}|^2$;

(iv) By considering the triangle *ATB*, show that the length of *OT* is given by $(\mathbf{a}\cdot\mathbf{b})^{\frac{1}{2}}$

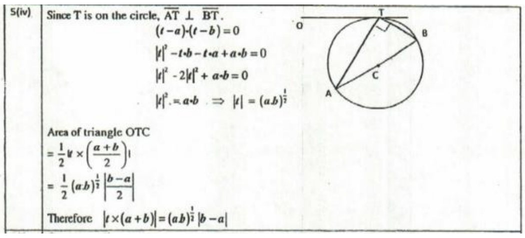
By considering the area of triangle *OTC*, show that $|\mathbf{t}\times(\mathbf{a}+\mathbf{b})| = (\mathbf{a}\cdot\mathbf{b})^{\frac{1}{2}}\,|\mathbf{b}-\mathbf{a}|$

Ans: (i) $\overline{OC} = \frac{\mathbf{a}+\mathbf{b}}{2}$

Answer



5(i)	(i) $\overline{OC} = \frac{a+b}{2}$
5(ii)	Let the angle between the vectors $\overline{OA}$ and $\overline{OB}$ be θ. $a\cdot b > 0 \Rightarrow |a||b|\cos\theta > 0$ $\Rightarrow \cos\theta > 0$ $\Rightarrow 0 < \theta < 90°$ Therefore angle AOB is acute. Since for any point P on or inside the circle with AB as diameter, angle APB $\geq 90°$, hence O must be outside the circle.
5(iii)	Since T is a tangent to the circle, $\overline{OT} \perp \overline{CT}$. $t\cdot\left(t-\frac{a+b}{2}\right)=0$ $|t|^2 - t\cdot\left(\frac{a+b}{2}\right)=0$ $t\cdot\left(\frac{a+b}{2}\right)=|t|^2$ Alternative : $t\cdot c = |t||c|\cos\theta = |t|^2$ (as $|c|\cos\theta = |t|$)



5(iv)	Since T is on the circle, $\overline{AT} \perp \overline{BT}$. $(t-a)\cdot(t-b)=0$ $|t|^2 - t\cdot b - t\cdot a + a\cdot b = 0$ $|t|^2 - 2|t|^2 + a\cdot b = 0$ $|t|^2 = a\cdot b \Rightarrow |t| = (a.b)^{\frac{1}{2}}$ Area of triangle OTC $= \frac{1}{2}|t\times\left(\frac{a+b}{2}\right)|$ $= \frac{1}{2}(a.b)^{\frac{1}{2}}\left|\frac{b-a}{2}\right|$ Therefore $|t\times(a+b)| = (a.b)^{\frac{1}{2}}\,|b-a|$

Q3

Question

Planes P_1, P_2 and P_3 have equations

$$-2x + z = 4$$

$$2x + y - 2z = 6$$

$$-6x + 4y + \lambda z = \mu$$

Respectively, where λ and μ are constants.

(i) Find a vector parallel to both P_1 and P_2.
Given that the point with coordinates $(-5, \alpha, \beta)$, lies on P_1 and P_2, find α and β.
Hence find a vector equation of the line of intersection of P_1 and P_2.

(ii) Given that P_1, P_2 and P_3 form a triangular prism, what can be said about the values of λ and μ?

Ans: (i) $\begin{pmatrix}1\\2\\2\end{pmatrix}$, $\alpha = 4, \beta = -6$, $r = \begin{pmatrix}-5\\4\\-6\end{pmatrix} + \delta\begin{pmatrix}1\\2\\2\end{pmatrix}$ (ii)$\lambda = -1$, $\mu \neq 52$

Answer

5(i) The vector parallel to both p_1 and p_2

$$\begin{pmatrix}-2\\0\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}-1\\-2\\-2\end{pmatrix}=\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$\begin{pmatrix}-5\\\alpha\\\beta\end{pmatrix}\cdot\begin{pmatrix}-2\\0\\1\end{pmatrix}=4 \qquad \begin{pmatrix}-5\\\alpha\\\beta\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}=6$$

$10+\beta=4 \qquad -10+\alpha-2\beta=6$

$\beta=-6 \qquad \alpha=6+10+2(-6)=4$

The vector equation of the line in which p_1 and p_2 intersect is

$$r=\begin{pmatrix}-5\\4\\-6\end{pmatrix}+\delta\begin{pmatrix}1\\2\\2\end{pmatrix},\ \delta\in\mathbb{R}$$

5(ii)

Vector equation of the line in which p_1 and p_2 intersect must be perpendicular to the normal of p_3

$$\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}-6\\4\\\lambda\end{pmatrix}=0$$

$-6+8+2\lambda=0$

$\lambda=-1$

The point (-5, 4, -6) must not be on p_3

$$\mu\neq\begin{pmatrix}-5\\4\\-6\end{pmatrix}\cdot\begin{pmatrix}-6\\4\\1\end{pmatrix}=52$$

Q4

Question

The planes $P_1$and P_2 have equations $r.\begin{pmatrix}-1\\-2\\2\end{pmatrix}=-1$ and $r.\begin{pmatrix}-7\\4\\4\end{pmatrix}=1$ respectively.

(i) Find the acute angle between $P_1$and P_2.

(ii) The point $A(2\ ,\alpha\ ,3)$ is equidistant from the planes P_1 and P_2. Calculate the two possible values of α

(iii) Find the position vector of the foot of perpendicular from $B(0,1,2)$ to the plane P_1. Hence find the Cartesian equation of the plane P_3 such that P_3 is parallel to P_1 and point B is equidistant from planes $P_1$and P_3

Ans: (i) $74.97°$ (ii) $\alpha=\frac{9}{5}\ or\ 6$ (iii) position vector $\overline{OM}=\frac{1}{3}\begin{pmatrix}1\\5\\4\end{pmatrix}$, equation of plane $x-2y+2z=5$

Answer

8(i) The acute angle between p_1 and p_2

$$=\cos^{-1}\left|\frac{\begin{pmatrix}-7\\4\\4\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}}{\sqrt{7^2+4^2+4^2}\sqrt{1^2+2^2+2^2}}\right|$$

$=74.97° = 75.0°$

8(ii)

$\begin{pmatrix}1\\0\\0\end{pmatrix}$ is a point on p_1.

Let point D be $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}$

$=\begin{pmatrix}1\\0\\0\end{pmatrix}-\begin{pmatrix}2\\\alpha\\3\end{pmatrix}=\begin{pmatrix}-1\\-\alpha\\-3\end{pmatrix}$

Perpendicular distance from A to p_1

$$\left|\frac{\overrightarrow{AD}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}}{\sqrt{1^2+2^2+2^2}}\right|$$

$$=\left|\frac{\begin{pmatrix}-1\\-\alpha\\-3\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}}{\sqrt{1^2+2^2+2^2}}\right|$$

$=\left|\frac{1+2\alpha-6}{\sqrt{9}}\right|$

$=\left|\frac{-5+2\alpha}{3}\right|$

$\begin{pmatrix}1\\2\\0\end{pmatrix}$ is a point on p_2.

Let point C be $\begin{pmatrix}1\\2\\0\end{pmatrix}$.

$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$

$=\begin{pmatrix}1\\2\\0\end{pmatrix}-\begin{pmatrix}2\\\alpha\\3\end{pmatrix}=\begin{pmatrix}-1\\2-\alpha\\-3\end{pmatrix}$

Perpendicular distance from A to p_2

$$\left|\frac{\overrightarrow{AC}\cdot\begin{pmatrix}-7\\4\\4\end{pmatrix}}{\sqrt{7^2+4^2+4^2}}\right|$$

$$=\left|\frac{\begin{pmatrix}-1\\2-\alpha\\-3\end{pmatrix}\cdot\begin{pmatrix}-7\\4\\4\end{pmatrix}}{\sqrt{7^2+4^2+4^2}}\right|$$

$=\left|\frac{7+8-4\alpha-12}{\sqrt{81}}\right|$

$=\left|\frac{3-4\alpha}{9}\right|$

$\therefore\left|\frac{-5+2\alpha}{3}\right|=\left|\frac{3-4\alpha}{9}\right|$

$\frac{-5+2\alpha}{3}=\frac{3-4\alpha}{9}$

$-15+6\alpha=3-4\alpha$

$10\alpha=18$

$\alpha=\frac{18}{10}=\frac{9}{5}=1\frac{4}{5}$

or

$\frac{-5+2\alpha}{3}=-\frac{3-4\alpha}{9}$

$-15+6\alpha=-3+4\alpha$

$2\alpha=12$

$\alpha=\frac{12}{2}=6$

8(iii) Let M be the position vector of the foot of perpendicular from B to p_1.

Equation of line segment BM

$r=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\2\end{pmatrix}$

When line segment BM intersects p_1.

$\begin{pmatrix}-\lambda\\1-2\lambda\\2+2\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}=-1$

$\lambda-2+4\lambda+4+4\lambda=-1$

$\lambda=-\frac{1}{3}$

$\therefore\overrightarrow{OM}=\frac{1}{3}\begin{pmatrix}1\\5\\4\end{pmatrix}$

Let point M' be a point in plane p_3 such that $\overrightarrow{MB}=\overrightarrow{BM'}$.

$\overrightarrow{MB}=\overrightarrow{BM'}$

$\overrightarrow{OM'}=2\overrightarrow{OB}-\overrightarrow{OM}$

$\overrightarrow{OM'}=2\begin{pmatrix}0\\1\\2\end{pmatrix}-\frac{1}{3}\begin{pmatrix}1\\5\\4\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-1\\1\\8\end{pmatrix}$

Equation of plane p_3

$r.\begin{pmatrix}-1\\-2\\2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-1\\1\\8\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}=5$

$-x-2y+2z=5$

Q5

Question

The line l_1 passes through the point $(1, -1, 1)$ and is parallel to the vector $\mathbf{2i}-\mathbf{5j}+\mathbf{3k}$.

The line l_2 has equation $x-1=\frac{2y+4}{4}=-\frac{z}{2}$

Given that the plane Π_1 contains l_1 and is parallel to l_2

(i) Find the Cartesian equation of the plane Π_1

(ii) Find the shortest distance between Π_1 and l_2, leaving your answer in exact form.

The plane Π_2 has equation $\mathbf{r}.(\mathbf{3i}-\mathbf{2k})=\mathbf{1}$

(iii) Find the acute angle between the planes Π_1 and Π_2

(iv) Determine the geometrical relationship between Π_2 and l_1, showing your working clearly.

(v) Hence what can be said about the values of a and b such that there is no solution for the following system of linear equations?

$$4x+7y+9z=6$$
$$3x-2z=1$$
$$2x+y-az=b$$

Ans: (i) equation of plane $4x+7y+9z=6$ (ii) $\frac{16}{\sqrt{146}}$ (iii) 82.1° (iv) Π_2 contains l_1 (v) $a=-\frac{1}{3}, b\neq\frac{4}{3}$

Answer

9(i)	The vector equation of l_2 is $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ The normal vector of $\Pi_1 = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix}$ Scalar Product form of equation of Π_1 is $\mathbf{r} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} = 6$ Cartesian equation of the plane Π_1 is $4x + 7y + 9z = 6$
(ii)	Let d be the shortest distance. Then $d = \dfrac{\left\lvert \overrightarrow{AB} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} \right\rvert}{\sqrt{9^2 + 7^2 + 4^2}}$, where $A = (1, -1, 1)$ and $B(1, -2, 0)$ $= \dfrac{\left\lvert \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} \right\rvert}{\sqrt{146}}$ $= \dfrac{16}{\sqrt{146}}$
(iii)	Let θ be the angle between the two planes. $\cos\theta = \dfrac{\left\lvert \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} \right\rvert}{\sqrt{13}\sqrt{146}} = \dfrac{6}{\sqrt{13}\sqrt{146}}$ $\theta = 82.1°$
(iv)	Note that (1, -1, 1) lies on Π_2 since $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = 1$ Also, note that l_1 is parallel to Π_2 since $\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = 0$ So Π_2 contains l_1.
(v)	Since both Π_1 and Π_2 contain l_1, l_1 is parallel to the plane $2x + y - az = b$ So $\begin{pmatrix} 2 \\ 1 \\ -a \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = 0$ $\Rightarrow a = -\dfrac{1}{3}$ Note that (1, -1, 1) does not lie on plane $2x + y - az = b$ So $2(1) - 1 - a(1) \neq b \Rightarrow b \neq \dfrac{4}{3}$.

Q6

Question

Do not use a calculator in answering this question.

Relative to the origin O, two points A and B have position vectors given by $\mathbf{a} = p\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

and $\mathbf{b} = \mathbf{i}$ respectively.

(i) The point *C* is on *AB* such that $AC: CB = 2: 1$. Find the position vector of *C* in terms of *p*. Hence find the exact area of triangle *OAC.*

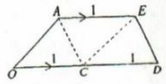
(ii) The point *D* is on *OC* produced such that $OD = 2CD$. The point *E* is such that $\overrightarrow{AE} = \overrightarrow{OC}$. Find the area of trapezium *OAED*.

(iii) Given that the angle between $\mathbf{a}$ and $\mathbf{b}$ is 135°, find the value of *p*.

Ans: (i) $\overrightarrow{OC} = \frac{1}{3}\begin{pmatrix} p+2 \\ 1 \\ -3 \end{pmatrix}$, area = $\frac{\sqrt{10}}{3}$ (ii) area = $\sqrt{10}$ (iii) $p = -\sqrt{10}$

Answer

Qn	Suggested Solution
4(i)	 $\overrightarrow{OC}=\dfrac{\mathbf{a}+2\mathbf{b}}{3}=\dfrac{\begin{pmatrix}p\\1\\-3\end{pmatrix}+2\begin{pmatrix}1\\0\\0\end{pmatrix}}{3}=\dfrac{1}{3}\begin{pmatrix}p+2\\1\\-3\end{pmatrix}$ Area of triangle OAC $=\dfrac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OC}\right\|$ $=\dfrac{1}{2}\left\|\begin{pmatrix}p\\1\\-3\end{pmatrix}\times\dfrac{1}{3}\begin{pmatrix}p+2\\1\\-3\end{pmatrix}\right\|$ $=\dfrac{1}{6}\left\|\begin{pmatrix}0\\-6-3p+3p\\p-2-p\end{pmatrix}\right\|=\dfrac{2}{6}\left\|\begin{pmatrix}0\\-3\\-1\end{pmatrix}\right\|$ $=\dfrac{\sqrt{10}}{3}$

Qn	Suggested Solution
4(ii)	 Triangle OAD, triangle ADE and triangle OAC have the same height and base and thus they have the same area. Area of trapezium $OAED$ $=3\left(\dfrac{\sqrt{10}}{3}\right)=\sqrt{10}$
4(iii)	$\cos 135^\circ=\dfrac{\begin{pmatrix}p\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\sqrt{p^2+10}\sqrt{1}}$ $-\dfrac{1}{\sqrt{2}}=\dfrac{p}{\sqrt{p^2+10}}$ $p^2+10=2p^2$ $p^2=10$ $p=-\sqrt{10}$ (reject $p=\sqrt{10}$ since $\mathbf{a}\cdot\mathbf{b}<0$)

Q7

Question

The line l has equation $\frac{x-2}{-1}=\frac{z-a}{1}$, $y=-1$, where a is a real constant and the plane p_1 has equation $3x+y+2z=5$. The point A has position vector $\mathbf{2i}+\mathbf{2j}$ with respect to the origin *O*.

(i) Find the acute angle between l and p_1.

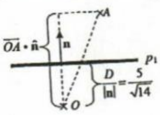
(ii) Find the perpendicular distance from the point A and p_1

(iii) Given that l is the line of intersection of the planes p_2 and p_3 with equations $x-4y+z=6$ and $x-y+bz=c$, where b and c are real constants. Find b and c.

(iv) The point B varies such that the midpoint of AB is always in p_1. Find a cartesian equation for the locus of B.

Ans: (i) $10.9°$ (ii) $\frac{3}{\sqrt{14}}$ (iii) $b=1, c=3$ (iv) $3x+y+2z=2$

Answer

Qn	Suggested Solution
4(i)	$l: \underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ a \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ Let the acute angle between l and p_1 be θ. $\cos(90^\circ - \theta) = \sin\theta = \left\lvert \dfrac{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{2}\sqrt{14}} \right\rvert = \dfrac{1}{\sqrt{28}}$ $\theta = 10.9^\circ$
4(ii)	Method 1 $(0,5,0)$ is a point on p_1. Perpendicular distance from the point A to p_1 $= \dfrac{\left\lvert \left[\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\rvert}{\sqrt{14}}$ $= \dfrac{\left\lvert \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\rvert}{\sqrt{14}} = \dfrac{3}{\sqrt{14}}$ Method 2 Perpendicular distance from the point A to p_1 $= \dfrac{\left\lvert 5 - \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\rvert}{\sqrt{14}}$ $= \dfrac{\lvert 5-8 \rvert}{\sqrt{14}} = \dfrac{3}{\sqrt{14}}$  Method 3 Let F be the foot of perpendicular and it lies on both l_{AF} and p_1. $l_{AF}: \underset{\sim}{r} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \ \mu \in \mathbb{R}$ $\left[\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 5$ $\Rightarrow \mu = -\dfrac{3}{14}$ $\overline{OF} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \left[-\dfrac{3}{14} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right]$ $\overline{AF} = \overline{OF} - \overline{OA} = -\dfrac{3}{14} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ $\lvert \overline{AF} \rvert = \left\lvert -\dfrac{3}{14} \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \right\rvert = \dfrac{3\sqrt{14}}{14}$
4(iii)	$l: \underset{\sim}{r} = \begin{pmatrix} 2 \\ -1 \\ a \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ **To find b:** Method 1 Direction vector of l is perpendicular to normal vector of p_3, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} = 0 \Rightarrow -1 + b = 0 \Rightarrow b = 1$ Method 2 $\begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} = k \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1-4b \\ 1-b \\ 3 \end{pmatrix} = k \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow k = 3, \ b = 1$ **To find c:** Method 1 Since l lies on p_2, $\begin{pmatrix} 2 \\ -1 \\ a \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} = 6$ $2 - 4(-1) + a = 6 \Rightarrow a = 0$ Since l lies on p_3, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = c$ $2 - (-1) + b(0) = c \Rightarrow c = 3$ Method 2 Since l lies on p_2, $\begin{pmatrix} 2-\lambda \\ -1 \\ a+\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} = 6$ $\Rightarrow 6 + a = 6 \Rightarrow a = 0$ Since l lies on p_3, $\begin{pmatrix} 2-\lambda \\ -1 \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ b \end{pmatrix} = c$ $(3-c) + (b-1)\lambda = 0$ Since the equation is always true regardless of λ, $3 - c = 0 \Rightarrow c = 3$ & $b - 1 = 0 \Rightarrow b = 1$
4(iv)	Let M be the midpoint of A and B. $\overline{OM} = \dfrac{\overline{OA} + \overline{OB}}{2} = \dfrac{1}{2}\left[\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \overline{OB} \right]$ $\dfrac{1}{2}\left[\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \overline{OB} \right] \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 5$ $\Rightarrow \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} + \overline{OB} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 10$ $\Rightarrow \overline{OB} \cdot \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 10 - 8 = 2$ $\therefore$ A cartesian equation for the locus of B is $3x + y + 2z = 2$. Note: Locus is a plane parallel to p_1

Q8

Question

Relative to the origin *O*, the position vectors of points *A* and *B* are **a** and **b** respectively, where **a** and **b** are non-zero and non-parallel vectors. The point *P* on *OA* is such that $OP : PA = 2:3$. The point *Q* is such that *OPQB* is a parallelogram.

(i) Find $\overrightarrow{OQ}$ in terms of **a** and **b**

(ii) Show that the area of the triangle *OAQ* can be written as $k|\mathbf{a} \times \mathbf{b}|$, where *k* is a constant to be found.

(iii) State the ratio of the area of triangle *OPB* to area of triangle *OAB*.

(iv) Given $\mathbf{a} \times \mathbf{b}$ is a unit vector, $|\mathbf{a}| = 2$ and the angle between **a** and **b** is 60°, find the exact value of $|\mathbf{b}|$

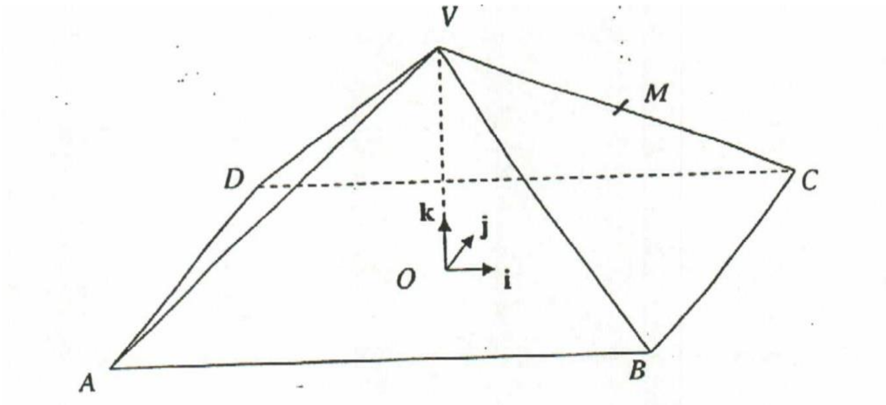
Ans: (i) $\overrightarrow{OQ} = \frac{2}{5}\mathbf{a} + \mathbf{b}$ (ii) $k = \frac{1}{2}$ (iii) $2:5$ (iv) $|\mathbf{b}| = \frac{1}{\sqrt{3}}$

Answer

9(i)	$\overline{OP} = \frac{2}{5}\mathbf{a}$ Since *OPQB* is a parallelogram, $\overline{OB} = \overline{PQ}$ $\mathbf{b} = \overline{OQ} - \overline{OP}$ $\mathbf{b} = \overline{OQ} - \frac{2}{5}\mathbf{a}$ $\overline{OQ} = \frac{2}{5}\mathbf{a} + \mathbf{b}$
(ii)	Area of triangle *OAQ* $= \frac{1}{2}\lvert \overline{OA} \times \overline{OQ} \rvert$ $= \frac{1}{2}\left\lvert \mathbf{a} \times \left(\frac{2}{5}\mathbf{a} + \mathbf{b} \right) \right\rvert$ $= \frac{1}{2}\left\lvert \mathbf{a} \times \frac{2}{5}\mathbf{a} + \mathbf{a} \times \mathbf{b} \right\rvert$ $= \frac{1}{2}\lvert \mathbf{a} \times \mathbf{b} \rvert$ Therefore, k is $\frac{1}{2}$.
(iii)	$OPB : OAB$ $2:5$
(iv)	Since $\mathbf{a} \times \mathbf{b}$ is a unit vector, $\lvert \mathbf{a} \times \mathbf{b} \rvert = \lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \sin\theta$ $1 = \lvert \mathbf{a} \rvert \lvert \mathbf{b} \rvert \sin\theta$ $1 = 2\lvert \mathbf{b} \rvert \sin 60^\circ$ Therefore, $\lvert \mathbf{b} \rvert = \frac{1}{\sqrt{3}}$

Q9

Question



In the diagram, O is centre of the rectangular base $ABCD$ of a right pyramid with vertex V. Perpendicular unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are parallel to AB, BC, OV respectively. The length of AB, BC, OV are $12, 6$, and 6 units respectively. The point M is the mid-point of CV and the point O is taken as the origin for position vectors.

(i) Show that the equation of the line AM may be expressed as $\boldsymbol{r} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$, where t is a parameter.

(ii) Find the perpendicular distance from B to the line AM.

(iii) Find the acute angle between the line DV and the plane AMB.

The plane Π has equation $\boldsymbol{r}.\begin{pmatrix} -1 \\ 4 \\ a \end{pmatrix} = 4$

(iv) Given that the three planes AMB, AMD and Π have no point in common, find the value of a.

Ans: (ii) 6.18 (iii) $47.7°$ (iv) $a = -3$

Answer

4(i)	$\overrightarrow{OA} = -6\underset{\sim}{i} - 3\underset{\sim}{j} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix}, \overrightarrow{OC} = -\overrightarrow{OA} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}, \overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$ $\therefore \overrightarrow{OM} = \frac{1}{2}\left[\begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}\right] = \begin{pmatrix} 3 \\ \frac{3}{2} \\ 3 \end{pmatrix}$ $\therefore \overrightarrow{AM} = \begin{pmatrix} 3 \\ \frac{3}{2} \\ 3 \end{pmatrix} - \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ \frac{9}{2} \\ 3 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$ Hence, equation of the line AM is $\mathbf{r} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$, $t \in \mathbb{R}$

(ii)

$$\overrightarrow{AB}=\begin{pmatrix}12\\0\\0\end{pmatrix}$$

Length of projection of $\overrightarrow{AB}$ onto the line AM,

$$AN=\frac{\left|\overrightarrow{AB}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}\right|}{\sqrt{6^2+3^2+2^2}}$$

$$=\frac{\left|\begin{pmatrix}12\\0\\0\end{pmatrix}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}\right|}{7}=\frac{72}{7}$$

Perpendicular distance from P to the line AM

$$=\sqrt{(AB)^2-(AN)^2}$$

$$=\sqrt{12^2-\left(\frac{72}{7}\right)^2}=6.18\ (3\text{ s.f.})$$

<u>Alternative Method</u>

Perpendicular distance from P to the line AM

$$=\frac{\left|\overrightarrow{AB}\times\begin{pmatrix}6\\3\\2\end{pmatrix}\right|}{\sqrt{6^2+3^2+2^2}}=\frac{\left|\begin{pmatrix}12\\0\\0\end{pmatrix}\times\begin{pmatrix}6\\3\\2\end{pmatrix}\right|}{\sqrt{49}}$$

$$=\frac{12\left|\begin{pmatrix}0\\-2\\3\end{pmatrix}\right|}{7}=\frac{12\sqrt{4+9}}{7}$$

$$=6.18\ (3\text{ s.f.})$$

(iii) A normal vector to the plane AMB

$$=\begin{pmatrix}12\\0\\0\end{pmatrix}\times\begin{pmatrix}6\\3\\2\end{pmatrix}=12\begin{pmatrix}0\\-2\\3\end{pmatrix}$$

$$\overrightarrow{DV}=\begin{pmatrix}0\\0\\6\end{pmatrix}-\begin{pmatrix}-6\\3\\0\end{pmatrix}=\begin{pmatrix}6\\-3\\6\end{pmatrix}=3\begin{pmatrix}2\\-1\\2\end{pmatrix}$$

Let θ be that angle betw. the line DV and plane AMB.

$$\sin\theta=\frac{\left|\begin{pmatrix}2\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}0\\-2\\3\end{pmatrix}\right|}{\sqrt{4+1+4}\sqrt{4+9}}$$

$$=\frac{2+6}{3\sqrt{13}}$$

$$\theta=47.7^\circ\ (1\text{ d.p.})$$

(iv) If the 3 planes AMB, AMD and Π do not have a common point, the line AM is parallel to Π but does not lie in Π.

$$\therefore\ \begin{pmatrix}6\\3\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\a\end{pmatrix}=0$$

$$\Rightarrow -6+12+2a=0$$

$$\Rightarrow a=-3$$

Note that $\begin{pmatrix}-6\\-3\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\a\end{pmatrix}=6-12\neq 4$.

Therefore point A does not lie in Π.
Hence the line AM does not lie in Π.

Q10

Question

The plane π_1 has equation, $\boldsymbol{r}.\begin{pmatrix}\alpha\\\beta\\0\end{pmatrix}=-30$, where α and β are positive constants, and contains the point A with coordinates $(-10,0,5)$

(i) Given that the perpendicular distance from the origin O to the plane π_1 is 6, find α and β

Another plane π_2 has equation $\boldsymbol{r}.\begin{pmatrix}-1\\1\\2\end{pmatrix}=4$

(ii) Find the acute angle between the line OA and the plane π_2

(iii) Find a Cartesian equation of the plane π_3 which contains the line OA and is perpendicular to the plane π_2

Ans: (i) $\alpha=3,\beta=4$ (ii) $46.9°$ (iii) $x-3y+2z=0$

Answer

3	(i)	Since A lies in the plane π_1, $\begin{pmatrix}-10\\0\\5\end{pmatrix}\cdot\begin{pmatrix}\alpha\\\beta\\0\end{pmatrix}=-30$. $\Rightarrow -10\alpha=-30$ $\Rightarrow \alpha=3$
		Perpendicular distance from O to $\pi_1=6$ $\Rightarrow \frac{\lvert\overrightarrow{OA}\cdot\mathbf{n}_1\rvert}{\lvert\mathbf{n}_1\rvert}=6$ $\Rightarrow \frac{\lvert-30\rvert}{\lvert\mathbf{n}_1\rvert}=6$ $\Rightarrow \lvert\mathbf{n}_1\rvert=5$ $\Rightarrow \alpha^2+\beta^2=5^2$ $\Rightarrow \beta=4$

(ii)	Acute angle between OA and π_2 $=\sin^{-1}\frac{\lvert\overrightarrow{OA}\cdot\mathbf{n}_2\rvert}{\lvert\overrightarrow{OA}\rvert\lvert\mathbf{n}_2\rvert}$ $=\sin^{-1}\frac{\left\lvert\begin{pmatrix}-10\\0\\5\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\2\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}-10\\0\\5\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}-1\\1\\2\end{pmatrix}\right\rvert}$ $=\sin^{-1}\frac{20}{\sqrt{125}\sqrt{6}}$ $=46.9^\circ$
(iii)	$\begin{pmatrix}-10\\0\\5\end{pmatrix}\times\begin{pmatrix}-1\\1\\2\end{pmatrix}=\begin{pmatrix}-5\\15\\-10\end{pmatrix}$ A normal to plane π_3 is $\begin{pmatrix}1\\-3\\2\end{pmatrix}$. A cartesian equation is $x-3y+2z=0$.

Q11

Question

Referred to the origin O, the points A and B have position vectors **a** and **b** respectively. It is given that **a** and **b** are perpendicular to each other and have the same magnitude of 3 units each. Given that A, B and C are collinear.

(i) Show that **c** can be expressed as $\mathbf{c}=k\mathbf{b}+(1-k)\mathbf{a}$, where k is a constant

(ii) Find $|\mathbf{a}\times\mathbf{c}|$, in terms of k, and state its geometrical meaning.

(iii) It is given that the area of triangle OAC is three times the area of triangle OAB. Find the two possible values of k.
Given also that the length of projection of OC onto OA is 12 units, find **c** in terms of **a** and **b**.

Ans: (ii) $|\mathbf{a}\times\mathbf{c}|=9k$, area of parallelogram with sides OA and OC (iii) $k=\pm3$, $\mathbf{c}=-3\mathbf{b}+4\mathbf{a}$

Answer

10 (i)	Given A, B and C are collinear, $\overrightarrow{AC}=k\overrightarrow{AB}$ $\mathbf{c}-\mathbf{a}=k(\mathbf{b}-\mathbf{a})$ $\mathbf{c}=k\mathbf{b}+(1-k)\mathbf{a}$ (shown)
(ii)	$\lvert\mathbf{a}\times\mathbf{c}\rvert=\lvert\mathbf{a}\times[k\mathbf{b}+(1-k)\mathbf{a}]\rvert$ $=\lvert k(\mathbf{a}\times\mathbf{b})+(1-k)(\mathbf{a}\times\mathbf{a})\rvert$ $=\lvert k\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 90^\circ\hat{\mathbf{n}}+(1-k)\mathbf{0}\rvert$ $=9\lvert k\rvert$ It is the area of a parallelogram with sides OA and OC.
(iii)	Area of triangle $OAC=3\times$ area of triangle OAB $\frac{1}{2}\lvert\mathbf{a}\times\mathbf{c}\rvert=\frac{3}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert$ $9\lvert k\rvert=3\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin 90^\circ$ $=27$ $\lvert k\rvert=3$ $k=\pm3$.
(iv)	Length of projection of OC onto $OA=12$ $\frac{\lvert\mathbf{c}\cdot\mathbf{a}\rvert}{\lvert\mathbf{a}\rvert}=12$ $\lvert\mathbf{c}\cdot\mathbf{a}\rvert=12\lvert\mathbf{a}\rvert$ $=36$
	When $k=3$, $\mathbf{c}=3\mathbf{b}-2\mathbf{a}$ $\lvert\mathbf{c}\cdot\mathbf{a}\rvert=\lvert(3\mathbf{b}-2\mathbf{a})\cdot\mathbf{a}\rvert$ $=\lvert3(\mathbf{b}\cdot\mathbf{a})-2\mathbf{a}\cdot\mathbf{a}\rvert$ $=\lvert3(0)-2(3)^2\rvert$ $=18$
	When $k=-3$, $\mathbf{c}=-3\mathbf{b}+4\mathbf{a}$ $\lvert\mathbf{c}\cdot\mathbf{a}\rvert=\lvert(-3\mathbf{b}+4\mathbf{a})\cdot\mathbf{a}\rvert$ $=\lvert-3(\mathbf{b}\cdot\mathbf{a})+4\mathbf{a}\cdot\mathbf{a}\rvert$ $=\lvert-3(0)+4(3)^2\rvert$ $=36$
	$\therefore \mathbf{c}=-3\mathbf{b}+4\mathbf{a}$

Q12

Question

Referred to the origin O, the position vectors of the points A and C are **a** and **c** respectively, and *OABC* is a parallelogram.

Express $\overrightarrow{OB}$ and $\overrightarrow{AC}$ in terms of **a** and **c.**

Prove that $|\overrightarrow{OB}|^2 + |\overrightarrow{AC}|^2 = |\overrightarrow{OA}|^2 + |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{CO}|^2$

Using the result above, what can be said about **a** and **c** when $|\overrightarrow{OB}| = |\overrightarrow{AC}|$?

Ans: **a** is perpendicular to **c**

Answer

4	$\overline{OB} = \mathbf{a}+\mathbf{c}$ and $\overline{AC} = \mathbf{c}-\mathbf{a}$ $\|\overline{OB}\|^2 + \|\overline{AC}\|^2 = \|\mathbf{a}+\mathbf{c}\|^2 + \|\mathbf{c}-\mathbf{a}\|^2$ $= (\mathbf{a}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{c}) + (\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})$ $= \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{c} + \|\mathbf{c}\|^2 + \|\mathbf{a}\|^2 - 2\mathbf{a}\cdot\mathbf{c} + \|\mathbf{c}\|^2$ $= \|\overline{OA}\|^2 + \|\overline{AB}\|^2 + \|\overline{BC}\|^2 + \|\overline{CO}\|^2$ **Alternative solution:** $\|\overline{OA}\|^2 + \|\overline{AB}\|^2 + \|\overline{BC}\|^2 + \|\overline{CO}\|^2 = \|\mathbf{a}\|^2 + \|\mathbf{c}\|^2 + \|\mathbf{a}\|^2 + \|\mathbf{c}\|^2$ $= \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{c} + \|\mathbf{c}\|^2 + \|\mathbf{a}\|^2 - 2\mathbf{a}\cdot\mathbf{c} + \|\mathbf{c}\|^2$ $= (\mathbf{a}+\mathbf{c})\cdot(\mathbf{a}+\mathbf{c}) + (\mathbf{c}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})$ $= \|\mathbf{a}+\mathbf{c}\|^2 + \|\mathbf{c}-\mathbf{a}\|^2$ $= \|\overline{OB}\|^2 + \|\overline{AC}\|^2$
4	$\|\overline{OA}\|^2 + \|\overline{AB}\|^2 + \|\overline{BC}\|^2 + \|\overline{CO}\|^2 = \|\overline{OB}\|^2 + \|\overline{AC}\|^2$ $\|\overline{OA}\|^2 + \|\overline{OC}\|^2 = \|\overline{OB}\|^2$ $\|\mathbf{a}\|^2 + \|\mathbf{c}\|^2 = \|\mathbf{a}+\mathbf{c}\|^2$ $= \|\mathbf{a}\|^2 + 2\mathbf{a}\cdot\mathbf{c} + \|\mathbf{c}\|^2$ $\mathbf{a}\cdot\mathbf{c} = 0$ **a** is perpendicular to **c**.

Q13

Question

Referred to an origin *O*, the position vector of *A* is **2i – k** and the equation of line l is $r = -7\mathbf{i} + 15\mathbf{j} - 5\mathbf{k} + \lambda(3\mathbf{i} - 7\mathbf{j} + 4\mathbf{k})$.

(i) Find the position vectors of B and C, both lying on l, such that $AB = AC = 10$.

(ii) Given that M is the midpoint of BC and the plane π_1 contains l and is perpendicular to AM, show that the equation of the plane π_1 is $-3x + y + 4z = 16$.

(iii) The planes π_2 and π_3 have equations $x + 2y - 3z = 5$ and $x - 2y + z = 1$ respectively. Verify that A lies in both π_2 and π_3 .

(iv) Determine the position vector of D, the point of intersection between π_1, π_2 and π_3.

(v) Hence, or otherwise, find the volume of the tetrahedron $ABCD$. [Volume of tetrahedron = $\frac{1}{3}$ × area of triangular base × perpendicular height]

Ans: (i) $\overline{OB} = \begin{pmatrix} 2 \\ -6 \\ 7 \end{pmatrix}$, $\overline{OC} = \begin{pmatrix} -4 \\ 8 \\ -1 \end{pmatrix}$ (iv) $\overline{OD} = \begin{pmatrix} 15 \\ 13 \\ 12 \end{pmatrix}$ (v) $\frac{962}{3}$

Answer

12i	$\overline{AB}=\begin{pmatrix}-9+3\lambda\\15-7\lambda\\-4+4\lambda\end{pmatrix}$ $\lvert\overline{AB}\rvert^2=(-9+3\lambda)^2+(15-7\lambda)^2+(-4+4\lambda)^2=10^2$ $\lambda^2-4\lambda+3=0$ $\lambda=3$ or $\lambda=1$ $\overline{OB}=\begin{pmatrix}2\\-6\\7\end{pmatrix}$ $\overline{OC}=\begin{pmatrix}-4\\8\\-1\end{pmatrix}$
12ii	$\overline{OM}=\frac{1}{2}\left[\begin{pmatrix}2\\-6\\7\end{pmatrix}+\begin{pmatrix}-4\\8\\-1\end{pmatrix}\right]=\begin{pmatrix}-1\\1\\3\end{pmatrix}$ $\mathbf{n}=\overline{AM}=\begin{pmatrix}-1\\1\\3\end{pmatrix}-\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}-3\\1\\4\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}-3\\1\\4\end{pmatrix}=\begin{pmatrix}-7\\15\\-5\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\4\end{pmatrix}$ $-3x+y+4z=21+15-20$ $=16$
12iii	Since $\begin{pmatrix}2\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-3\end{pmatrix}=5$ and $\begin{pmatrix}2\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=1$, A lies in both π_1 and π_2.
12iv	$-3x+y+4z=16$ $x+2y-3z=5$ $x-2y+z=1$ Using GC, $x=15$, $y=13$, $z=12$ $\overline{OD}=\begin{pmatrix}15\\13\\12\end{pmatrix}$
12v	perpendicular height $=\lvert\overline{AM}\rvert$ $=\sqrt{9+1+16}$ $=\sqrt{26}$ $\overline{BD}=\begin{pmatrix}13\\19\\5\end{pmatrix}$, $\overline{CD}=\begin{pmatrix}19\\5\\13\end{pmatrix}$ area of base $=\frac{1}{2}\lvert\overline{BD}\times\overline{CD}\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}222\\-74\\-296\end{pmatrix}\right\rvert$ $=\sqrt{35594}$ volume $=\frac{1}{3}\times\sqrt{26}\times\sqrt{35594}$ $=\frac{962}{3}$ units^3

Q14

Question

Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. It is given that $|\mathbf{a}|=3$, $|\mathbf{b}|=5$ and $|3\mathbf{a}-\mathbf{b}|=10$.

(i) Give the geometrical interpretation of $\left|\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right|$.

(ii) Show that $\mathbf{a}\cdot\mathbf{b}=1$

(iii) Hence find the shortest distance from A to the line OB, and the area of the triangle OAB.

(iv) Given that $\mathbf{a}$, $2\mathbf{a}+3\mathbf{b}$ and $\mu\mathbf{a}+2\mathbf{b}$, where μ is a constant, are position vectors of collinear points, find μ.

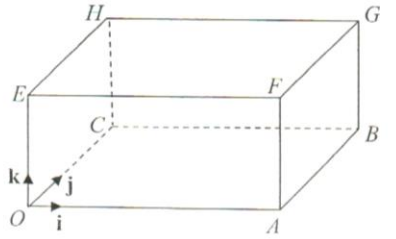
Ans: (i) length of projection of $\overline{OA}$ onto $\overline{OB}$ (iii) distance=$\frac{1}{5}$, area=7.48 (iv) $\mu=\frac{5}{3}$

Answer

Qn. [Marks]	Solution
6(i) [1]	$\left\lvert\mathbf{a}\cdot\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}\right\rvert$ represents the length of projection of $\overline{OA}$ onto $\overline{OB}$.
(ii) [2]	$\lvert 3\mathbf{a}-\mathbf{b}\rvert^2=10^2=100$ $9\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2-6\mathbf{a}\cdot\mathbf{b}=100$ $6\mathbf{a}\cdot\mathbf{b}=9(3)^2+(5)^2-100=6$ Therefore $\mathbf{a}\cdot\mathbf{b}=1$.
(iii) [2]	Let N be the foot of the perpendicular from A to the line OB. $ON=\left\lvert\mathbf{a}\cdot\frac{\mathbf{b}}{\lvert\mathbf{b}\rvert}\right\rvert=\frac{1}{5}$. Using Pythagoras Theorem, $AN^2=OA^2-ON^2=3^2-\left(\frac{1}{5}\right)^2=\frac{224}{25}$. $AN=\sqrt{\frac{224}{25}}=\frac{4}{5}\sqrt{14}=2.9933=2.99$ (3sf). Area of triangle $OAB=\frac{1}{2}OB\times AN=2\sqrt{14}=7.48$ (3sf).
(iv) [4]	$(\mu\mathbf{a}+2\mathbf{b})-\mathbf{a}=k[(2\mathbf{a}+3\mathbf{b})-\mathbf{a}]$ for some constant k. Now, $\lvert\mathbf{a}\rvert\neq 0$, $\lvert\mathbf{b}\rvert\neq 0$ and $\lvert\mathbf{a}\cdot\mathbf{b}\rvert=1\neq\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert$, so $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. $(\mu-1)\mathbf{a}+2\mathbf{b}=k\mathbf{a}+3k\mathbf{b}$ $(\mu-1-k)\mathbf{a}=(3k-2)\mathbf{b}$ Hence $3k-2=0\Rightarrow k=\frac{2}{3}$ and $\mu=k+1=\frac{5}{3}$.

Q15

Question:



The diagram shows a cuboid with rectangular base $OABC$ and top $EFGH$, where $OA = 4$ units, $OC = 3$ units and $OE = 2$ units. The point O is taken as the origin and unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, are taken along OA, OC and OE respectively.

(i) Find the cartesian equation of the plane p which contains the points A, C and E. [3]

(ii) Find the acute angle between p and the base $OABC$. [2]

The line l, passing through O, is perpendicular to p and intersects the plane containing B, C, G and H at the point T.

(iii) Find the position vector of the point T and deduce the perpendicular distance from T to p. [5]

(iv) A point Q lies on the line passing through C and T such that its distance from p is twice that of the distance from T to p. Find the possible position vectors of the point Q. [3]

Solution

12(i)

$\overrightarrow{AE} \times \overrightarrow{CE}$

$= \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$

Equation of p :

$\mathbf{r}.\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}.\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = 12$

Cartesian equation of p : $3x + 4y + 6z = 12$

(ii) Let the acute angle be θ.

$\cos\theta = \dfrac{\left|\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}.\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|}{\sqrt{3^2+4^2+6^2}} = \dfrac{6}{\sqrt{61}}$

$\therefore \theta = 39.8°$

(iii)

Equation of l : $\mathbf{r} = \lambda\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}, \lambda \in \mathbb{R}$

Equation of plane containing B, C, G and H

$\mathbf{r}.\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 3$

At the intersection,

$\lambda\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 3$

$\lambda = \dfrac{3}{4}$

$\overrightarrow{OT} = \dfrac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$

Perpendicular distance from T to $p = \dfrac{\left|\overrightarrow{AT}.\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}\right|}{\left|\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}\right|}$

$= \dfrac{\left|\overrightarrow{OT}.\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} - \overrightarrow{OA}.\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}\right|}{\sqrt{61}}$

$= \dfrac{\left|\dfrac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}.\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} - 12\right|}{\sqrt{61}}$

$= \dfrac{135}{244}\sqrt{61}$

Alternative

$\left|\overrightarrow{OT}\right| = \left|\dfrac{3}{4}\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}\right| = \dfrac{3}{4}\sqrt{61}$

Perpendicular distance from O to p

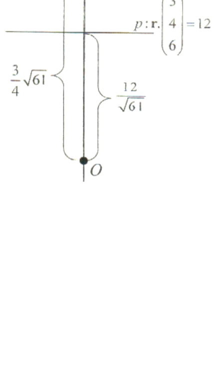
$= \dfrac{12}{\left|\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}\right|} = \dfrac{12}{\sqrt{61}}$

Perpendicular distance from T to p

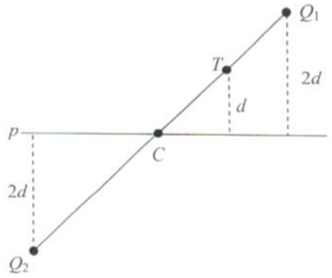
$= \dfrac{3}{4}\sqrt{61} - \dfrac{12}{\sqrt{61}}$

$= \dfrac{3}{4}\sqrt{61} - \dfrac{12}{61}\sqrt{61}$

$= \dfrac{135}{244}\sqrt{61}$



12(iv) Let the points be Q_1 and Q_2.



$\overrightarrow{CQ_1} = 2\overrightarrow{CT}$ and $\overrightarrow{CQ_2} = -2\overrightarrow{CT}$

$$\overrightarrow{CQ_1} = 2\left[\left(\frac{3}{4}\right)\begin{pmatrix}3\\4\\6\end{pmatrix} - \begin{pmatrix}0\\3\\0\end{pmatrix}\right]$$

$$\overrightarrow{OQ_1} - \overrightarrow{OC} = \frac{1}{2}\begin{pmatrix}9\\0\\18\end{pmatrix} \qquad \overrightarrow{OQ_2} - \overrightarrow{OC} = -\frac{1}{2}\begin{pmatrix}9\\0\\18\end{pmatrix}$$

$$\overrightarrow{OQ_1} = \frac{1}{2}\begin{pmatrix}9\\0\\18\end{pmatrix} + \begin{pmatrix}0\\3\\0\end{pmatrix} \qquad \overrightarrow{OQ_2} = -\frac{1}{2}\begin{pmatrix}9\\0\\18\end{pmatrix} + \begin{pmatrix}0\\3\\0\end{pmatrix}$$

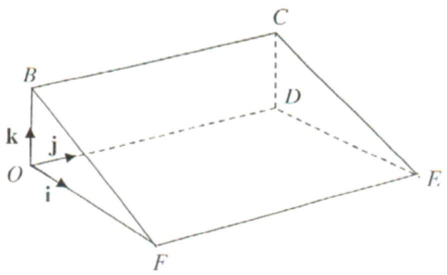
$$= \frac{1}{2}\begin{pmatrix}9\\6\\18\end{pmatrix} \qquad = \frac{1}{2}\begin{pmatrix}-9\\6\\-18\end{pmatrix}$$

The possible position vectors of the point Q are

$\frac{1}{2}\begin{pmatrix}9\\6\\18\end{pmatrix}$ and $\frac{1}{2}\begin{pmatrix}-9\\6\\-18\end{pmatrix}$

Q16

Question:



The diagram shows a vehicle ramp $OBCDEF$ with horizontal rectangular base $ODEF$ and vertical rectangular face $OBCD$. Taking the point O as the origin, the perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to the edges OF, OD and OB respectively. The lengths of OF, OD and OB are $2h$ units, 3 units and h units respectively.

(i) Show that $\overrightarrow{OC} = 3\mathbf{j} + h\mathbf{k}$. [1]

(ii) The point P divides the segment BC in the ratio $2:1$. Find $\overrightarrow{OP}$ in terms of h. [1]

(iii) A vector parallel to the normal of the plane $BCEF$ is given as $a\mathbf{i} + b\mathbf{k}$. By the use of a scalar product, find the value of $\frac{a}{b}$. Hence find the Cartesian equation of the plane $BCEF$ in terms of h. [4]

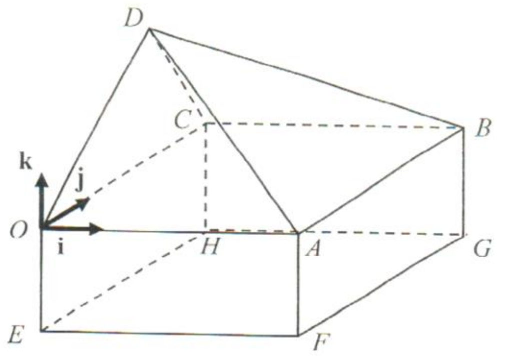
(iv) Take $h = 3$. Find the shortest distance from the point $Q(1, 2, 2)$ to the plane OPF. [4]

Solution:

7(i)	$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \begin{pmatrix}0\\0\\h\end{pmatrix} + \begin{pmatrix}0\\3\\0\end{pmatrix} = \begin{pmatrix}0\\3\\h\end{pmatrix}$
7(ii)	By ratio theorem $\overrightarrow{OP} = \dfrac{2\overrightarrow{OC} + \overrightarrow{OB}}{3}$ $= \dfrac{1}{3}\left[\begin{pmatrix}0\\6\\2h\end{pmatrix} + \begin{pmatrix}0\\0\\h\end{pmatrix}\right] = \begin{pmatrix}0\\2\\h\end{pmatrix}$
7(iii)	Select a suitable direction vector parallel to the plane such as $\overrightarrow{BE} = \overrightarrow{OE} - \overrightarrow{OB}$ $= \begin{pmatrix}2h\\3\\0\end{pmatrix} - \begin{pmatrix}0\\0\\h\end{pmatrix} = \begin{pmatrix}2h\\3\\-h\end{pmatrix}$. Thus $\overrightarrow{BE} \cdot (a\mathbf{i} + b\mathbf{k}) = 0$ $\Rightarrow \begin{pmatrix}2h\\3\\-h\end{pmatrix} \cdot \begin{pmatrix}a\\0\\b\end{pmatrix} = 0 \Rightarrow \dfrac{a}{b} = \dfrac{1}{2}$ Since C is on the plane, $\begin{pmatrix}0\\3\\h\end{pmatrix} \cdot \begin{pmatrix}1\\0\\2\end{pmatrix} = 2h$ $\Rightarrow \mathbf{r} \cdot \begin{pmatrix}1\\0\\2\end{pmatrix} = 2h \Rightarrow x + 2z = 2h$
7(iv)	Given that $h = 3$, $\overrightarrow{OP} = \begin{pmatrix}0\\2\\3\end{pmatrix}$, $\overrightarrow{OF} = \begin{pmatrix}6\\0\\0\end{pmatrix}$ $\mathbf{n} = \begin{pmatrix}0\\2\\3\end{pmatrix} \times \begin{pmatrix}6\\0\\0\end{pmatrix} = \begin{pmatrix}0\\18\\-12\end{pmatrix} = 6\begin{pmatrix}0\\3\\-2\end{pmatrix}$ The equation of plane OPF is $\mathbf{r} \cdot \begin{pmatrix}0\\3\\-2\end{pmatrix} = 0$ Shortest distance $= \dfrac{\left\lvert \begin{pmatrix}1\\2\\2\end{pmatrix} \cdot \begin{pmatrix}0\\3\\-2\end{pmatrix} \right\rvert}{\sqrt{0^2 + 3^2 + 2^2}} = \dfrac{2}{\sqrt{13}}$ units.

Q17

The diagram below shows a figure made up of a pyramid and a cuboid. The pyramid has a square base $OABC$ of side 6 units. The vertex D is 4 units vertically above R, the midpoint of OC. The cuboid shares the same square base and is of height 3 units.



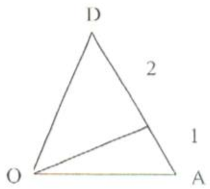
With O as the origin and using the unit vectors **i, j** and **k** given in the diagram,

(i) show that the position vector of point P is $4\mathbf{i}+\mathbf{j}+\frac{4}{3}\mathbf{k}$, where P lies on AD such that $AP:PD=1:2$, [3]

(ii) find the position vector of point Q in terms of **i, j** and **k**, where Q is the midpoint of FG. Hence, find the area of triangle OPQ. [4]

Solution:

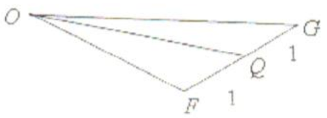
Solution



(i) $\overrightarrow{OD}=3\mathbf{j}+4\mathbf{k}$

$$\overrightarrow{OP}=\frac{2\overrightarrow{OA}+\overrightarrow{OD}}{3}$$
$$=\frac{2(6\mathbf{i})+3\mathbf{j}+4\mathbf{k}}{3}$$
$$=4\mathbf{i}+\mathbf{j}+\frac{4}{3}\mathbf{k}$$

(ii) $$\overrightarrow{OQ}=\frac{\overrightarrow{OF}+\overrightarrow{OG}}{2}$$
$$=\frac{(-3\mathbf{k}+6\mathbf{i})+(6\mathbf{i}+6\mathbf{j}-3\mathbf{k})}{2}$$
$$=\frac{12\mathbf{i}+6\mathbf{j}-6\mathbf{k}}{2}$$
$$=6\mathbf{i}+3\mathbf{j}-3\mathbf{k}$$



Area of triangle OPQ
$$=\frac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OQ}\right|$$
$$=\frac{1}{2}\left|\left(4\mathbf{i}+\mathbf{j}+\frac{4\mathbf{k}}{3}\right)\times(6\mathbf{i}+3\mathbf{j}-3\mathbf{k})\right|$$
$$=\frac{1}{2}\left|4\mathbf{i}\times6\mathbf{i}+4\mathbf{i}\times3\mathbf{j}-4\mathbf{i}\times3\mathbf{k}+\mathbf{j}\times6\mathbf{i}+\mathbf{j}\times3\mathbf{j}-\mathbf{j}\times3\mathbf{k}+\frac{4}{3}\mathbf{k}\times6\mathbf{i}+\frac{4}{3}\mathbf{k}\times3\mathbf{j}-\frac{4}{3}\mathbf{k}\times3\mathbf{k}\right|$$
$$=\frac{1}{2}\left|\mathbf{0}+12\mathbf{k}-12\mathbf{j}-6\mathbf{k}+\mathbf{0}-3\mathbf{i}-8\mathbf{j}-4\mathbf{i}-\mathbf{0}\right|$$
$$=\frac{1}{2}\sqrt{(-7)^2+20^2+6^2}$$
$$=11.0\text{ units}^2$$

OR

Area of triangle OPQ
$$=\frac{1}{2}\left|\overrightarrow{OP}\times\overrightarrow{OQ}\right|$$
$$=\frac{1}{2}\left|4\mathbf{i}+\frac{4\mathbf{k}}{3}\times6\mathbf{i}+3\mathbf{j}-3\mathbf{k}\right|$$
$$=\frac{1}{2}\left|\begin{pmatrix}4\\1\\\frac{4}{3}\end{pmatrix}\times\begin{pmatrix}6\\3\\-3\end{pmatrix}\right|$$
$$=\frac{1}{2}\left|\begin{pmatrix}-3-4\\-(-12-8)\\12-6\end{pmatrix}\right|$$
$$=\frac{1}{2}\sqrt{(-7)^2+(20)^2+6^2}$$
$$=11.0\text{ units}^2$$