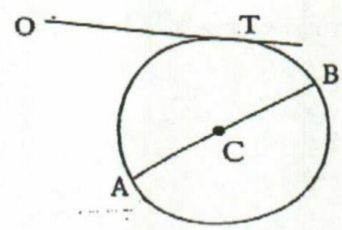
Q1

Question

Relative to an origin *O*, the points *A* and *B* have position vectors **a** and **b** respectively, where $\mathbf{a} \cdot \mathbf{b} > 0$. *A* and *B* also lie on a circle with centre *C* and *AB* as diameter.

(i) Write down the position vector of *C* in terms of **a** and **b**.
(ii) Show that the origin *O* is outside the circle.

T is a point on the circle with position vector **t** and *OT* is a tangent to the circle.

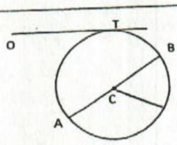
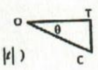


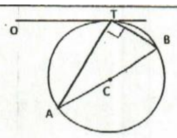
(iii) Show that $\mathbf{t} \cdot \left(\frac{\mathbf{a}+\mathbf{b}}{2}\right) = |\mathbf{t}|^2$;

(iv) By considering the triangle *ATB*, show that the length of *OT* is given by $(\mathbf{a} \cdot \mathbf{b})^{\frac{1}{2}}$

By considering the area of triangle *OTC*, show that $|\mathbf{t} \times (\mathbf{a}+\mathbf{b})| = (\mathbf{a} \cdot \mathbf{b})^{\frac{1}{2}}\,|\mathbf{b}-\mathbf{a}|$

Answer

5(i)	(i) $\overline{OC} = \frac{a+b}{2}$
5(ii)	Let the angle between the vectors $\overline{OA}$ and $\overline{OB}$ be θ. $a \cdot b > 0 \Rightarrow |a||b|\cos\theta > 0$ $\Rightarrow \cos\theta > 0$ $\Rightarrow 0 < \theta < 90°$ Therefore angle AOB is acute. Since for any point P on or inside the circle with AB as diameter, angle APB $\geq 90°$, hence O must be outside the circle.
5(iii)	Since T is a tangent to the circle, $\overline{OT} \perp \overline{CT}$. $t \cdot \left(t - \frac{a+b}{2}\right) = 0$ $|t|^2 - t \cdot \left(\frac{a+b}{2}\right) = 0$ $t \cdot \left(\frac{a+b}{2}\right) = |t|^2$   Alternative : $t \cdot c = |t||c|\cos\theta = |t|^2$ (as $|c|\cos\theta = |t|$)

5(iv)	Since T is on the circle, $\overline{AT} \perp \overline{BT}$. $(t-a)\cdot(t-b) = 0$ $|t|^2 - t \cdot b - t \cdot a + a \cdot b = 0$ $|t|^2 - 2|t|^2 + a \cdot b = 0$ $|t|^2 = a \cdot b \Rightarrow |t| = (a.b)^{\frac{1}{2}}$  Area of triangle OTC $= \frac{1}{2}|t \times \left(\frac{a+b}{2}\right)|$ $= \frac{1}{2}(a.b)^{\frac{1}{2}}\left|\frac{b-a}{2}\right|$ Therefore $|t \times (a+b)| = (a.b)^{\frac{1}{2}}\,|b-a|$

Q2

Question

Referred to the origin O, $\mathbf{a}$ and $\mathbf{b}$ are two non-zero and non-parallel vectors denoting the position vectors of the points *A* and *B* respectively.

If a point *C* has a position vector $\mathbf{c}$ such that $\mathbf{c} = (\mathbf{a}.\mathbf{a})\mathbf{a} + (\mathbf{a}.\mathbf{b})\mathbf{b}$,

(i) What can you say about the points O, A, B and C.

(ii) Given that $OBCA$ is a parallelogram, find $|\mathbf{a}|$.

The point U is the mid-point of OA and the point V on OB is such that $OV: VB = 3: 2$.

The point N lies on UV such that $UN: UV = 2: 3$.

(iii) Find $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$

(iv) If the angle AOB is $\frac{\pi}{3}$, find the area of triangle OUN, giving your answer in terms of $|\mathbf{b}|$

Answer

3(i)	*O*, *A*, *B* and *C* are coplanar **OR** *O*, *A*, *B* and *C* lie on the same plane.
(ii)	$\overline{AC} = \overline{OB}$ $\mathbf{c} - \mathbf{a} = \mathbf{b}$ $\mathbf{c} = \mathbf{a} + \mathbf{b}$ $(\mathbf{a}\cdot\mathbf{a})\mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\mathbf{b} = \mathbf{a} + \mathbf{b}$ $\Rightarrow |\mathbf{a}|^2 = 1$ and $\mathbf{a}\cdot\mathbf{b} = 1$ $\Rightarrow |\mathbf{a}| = 1$ So $\mathbf{a}$ is a unit vector.

(iii)	By ratio theorem, $\overline{ON} = \dfrac{\overline{OU} + 2\overline{OV}}{3}$ $= \frac{1}{3}\left(\frac{1}{2}\mathbf{a} + \frac{6}{5}\mathbf{b}\right) = \frac{1}{6}\mathbf{a} + \frac{2}{5}\mathbf{b}$
(iv)	Area of $OUN = \frac{1}{2}|\overline{OU} \times \overline{ON}|$ $= \frac{1}{2}\left|\frac{1}{2}a \times \left(\frac{1}{6}a + \frac{2}{5}b\right)\right|$ $= \frac{1}{2}\left|\frac{1}{12}(a \times a) + \frac{1}{5}(a \times b)\right|$ $= \frac{1}{10}|a \times b|$ $= \frac{1}{10}|\mathbf{a}||\mathbf{b}|\sin\frac{\pi}{3}$ $= \frac{1}{10}\left(\frac{\sqrt{3}}{2}\right)|\mathbf{b}|$ $= \frac{\sqrt{3}}{20}|\mathbf{b}|$

Q3

Question

Do not use a calculator in answering this question.

Relative to the origin O, two points A and B have position vectors given by $\mathbf{a} = p\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

and $\mathbf{b} = \mathbf{i}$ respectively.

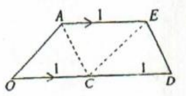
(i) The point *C* is on *AB* such that $AC: CB = 2: 1$. Find the position vector of *C* in terms of *p*. Hence find the exact area of triangle *OAC.*

(ii) The point *D* is on *OC* produced such that $OD = 2CD$. The point *E* is such that $\overrightarrow{AE} = \overrightarrow{OC}$. Find the area of trapezium *OAED*.

(iii) Given that the angle between $\mathbf{a}$ and $\mathbf{b}$ is 135°, find the value of *p*.

Answer

Qn	Suggested Solution
4(i)	A ——2—— C ——1—— B $\overrightarrow{OC}=\dfrac{\mathbf{a}+2\mathbf{b}}{3}=\dfrac{\begin{pmatrix}p\\1\\-3\end{pmatrix}+2\begin{pmatrix}1\\0\\0\end{pmatrix}}{3}=\dfrac{1}{3}\begin{pmatrix}p+2\\1\\-3\end{pmatrix}$ Area of triangle *OAC* $=\dfrac{1}{2}\lvert\overrightarrow{OA}\times\overrightarrow{OC}\rvert$ $=\dfrac{1}{2}\left\lvert\begin{pmatrix}p\\1\\-3\end{pmatrix}\times\dfrac{1}{3}\begin{pmatrix}p+2\\1\\-3\end{pmatrix}\right\rvert$ $=\dfrac{1}{6}\left\lvert\begin{pmatrix}0\\-6-3p+3p\\p-2-p\end{pmatrix}\right\rvert=\dfrac{2}{6}\left\lvert\begin{pmatrix}0\\-3\\-1\end{pmatrix}\right\rvert$ $=\dfrac{\sqrt{10}}{3}$

Qn	Suggested Solution
4(ii)	 Triangle *OAD*, triangle *ADE* and triangle *OAC* have the same height and base and thus they have the same area. Area of trapezium *OAED* $=3\left(\dfrac{\sqrt{10}}{3}\right)=\sqrt{10}$
4(iii)	$\cos 135^\circ=\dfrac{\begin{pmatrix}p\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}}{\sqrt{p^2+10}\sqrt{1}}$ $-\dfrac{1}{\sqrt{2}}=\dfrac{p}{\sqrt{p^2+10}}$ $p^2+10=2p^2$ $p^2=10$ $p=-\sqrt{10}$ (reject $p=\sqrt{10}$ since $\mathbf{a}\cdot\mathbf{b}<0$)

Q4

Question

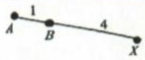
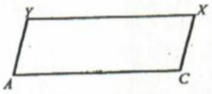
Relative to origin *O*, the points *A*, *B* and *C* have position vectors **a**, **b** and $\mathbf{a}+\mathbf{b}$ respectively. The point *X* is on *AB* produced such that $AB: AX$ is $1:5$ and the point *Y* is such that *ACXY* is a parallelogram. Given that the area of the triangle *OAB* is 1 square unit and **b** is a unit vector.

(i) Find in terms of **a** and **b**, the position vectors of *X* and *Y*. Hence show that *OABY* is a trapezium.

(ii) Give a geometrical meaning of $|(\mathbf{a}+\mathbf{b}).\mathbf{b}|$

(iii) Find the area of *ACXY*. Hence find the shortest distance from *X* to the line that passes through the points *A* and *C*.

Answer

8(i)	Using ratio theorem,  $\overrightarrow{OB}=\frac{4\overrightarrow{OA}+\overrightarrow{OX}}{5}$ $\Rightarrow \overrightarrow{OX}=5\overrightarrow{OB}-4\overrightarrow{OA}=5\mathbf{b}-4\mathbf{a}$  Since $ACXY$ forms a parallelogram, we have $\overrightarrow{AC}=\overrightarrow{YX}$ $\Rightarrow \overrightarrow{OY}=\overrightarrow{OX}-\overrightarrow{AC}$ $\Rightarrow \overrightarrow{OY}=(5\mathbf{b}-4\mathbf{a})-\mathbf{b}=4(\mathbf{b}-\mathbf{a})$ Since $\overrightarrow{OY}=4(\mathbf{b}-\mathbf{a}) \Rightarrow \overrightarrow{OY}=4\overrightarrow{AB}$ $\Rightarrow \overrightarrow{OY}$ is parallel to $\overrightarrow{AB} \Rightarrow OABY$ is a trapezium.
8(ii)	As $\mathbf{b}$ is a unit vector, thus $\lvert(\mathbf{a}+\mathbf{b})\cdot\mathbf{b}\rvert$ is the length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OB}$.
8(iii)	Area of $ACXY$ $=\lvert\overrightarrow{AC}\times\overrightarrow{AY}\rvert$ $=\lvert\mathbf{b}\times(4\mathbf{b}-5\mathbf{a})\rvert$ $=5\lvert\mathbf{b}\times\mathbf{a}\rvert$ $=5\times2=10$ [Area of $OAB=\frac{1}{2}\lvert\mathbf{b}\times\mathbf{a}\rvert=1$ $\Rightarrow \lvert\mathbf{b}\times\mathbf{a}\rvert=2$] Let the shortest distance be d. Then area of $ACXY$ = $AC\times d$ and $AC=OB=1$. Therefore, $d=10$.

Q5

Question

Relative to the origin *O*, the position vectors of points *A* and *B* are **a** and **b** respectively, where **a** and **b** are non-zero and non-parallel vectors. The point *P* on *OA* is such that $OP: PA = 2:3$. The point *Q* is such that *OPQB* is a parallelogram.

(i) Find $\overrightarrow{OQ}$ in terms of **a** and **b**

(ii) Show that the area of the triangle *OAQ* can be written as $k|\mathbf{a}\times\mathbf{b}|$, where *k* is a constant to be found.

(iii) State the ratio of the area of triangle *OPB* to area of triangle *OAB*.

(iv) Given $\mathbf{a}\times\mathbf{b}$ is a unit vector, $|\mathbf{a}| = 2$ and the angle between **a** and **b** is $60°$, find the exact value of $|\mathbf{b}|$

Answer

9(i)	$\overline{OP} = \frac{2}{5}\mathbf{a}$ Since *OPQB* is a parallelogram, $\overline{OB} = \overline{PQ}$ $\mathbf{b} = \overline{OQ} - \overline{OP}$ $\mathbf{b} = \overline{OQ} - \frac{2}{5}\mathbf{a}$ $\overline{OQ} = \frac{2}{5}\mathbf{a} + \mathbf{b}$
(ii)	Area of triangle *OAQ* $= \frac{1}{2}\left\|\overline{OA} \times \overline{OQ}\right\|$ $= \frac{1}{2}\left\|\mathbf{a} \times \left(\frac{2}{5}\mathbf{a} + \mathbf{b}\right)\right\|$ $= \frac{1}{2}\left\|\mathbf{a} \times \frac{2}{5}\mathbf{a} + \mathbf{a} \times \mathbf{b}\right\|$ $= \frac{1}{2}\left\|\mathbf{a} \times \mathbf{b}\right\|$ Therefore, k is $\frac{1}{2}$.
(iii)	*OPB* : *OAB* 2 : 5
(iv)	Since $\mathbf{a} \times \mathbf{b}$ is a unit vector, $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$ $1 = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$ $1 = 2\|\mathbf{b}\|\sin 60^\circ$ Therefore, $\|\mathbf{b}\| = \frac{1}{\sqrt{3}}$

Q6

Question

Referred to the origin O, the points *A* and *B* have position vectors **a** and **b** respectively. It is given that **a** and **b** are perpendicular to each other and have the same magnitude of 3 units each. Given that *A*, *B* and *C* are collinear.

(i) Show that **c** can be expressed as $\mathbf{c} = k\mathbf{b} + (1-k)\mathbf{a}$, where *k* is a constant

(ii) Find $|\mathbf{a} \times \mathbf{c}|$, in terms of *k*, and state its geometrical meaning.

(iii) It is given that the area of triangle *OAC* is three times the area of triangle *OAB*. Find the two possible values of *k*.
Given also that the length of projection of *OC* onto *OA* is 12 units, find **c** in terms of **a** and **b**.

Answer

10 **(i)**	Given A, B and C are collinear, $\overline{AC} = k\overline{AB}$ $\mathbf{c} - \mathbf{a} = k(\mathbf{b} - \mathbf{a})$ $\mathbf{c} = k\mathbf{b} + (1-k)\mathbf{a}$ (shown)	
(ii)	$\lvert \mathbf{a} \times \mathbf{c} \rvert = \lvert \mathbf{a} \times [k\mathbf{b} + (1-k)\mathbf{a}] \rvert$ $= \lvert k(\mathbf{a} \times \mathbf{b}) + (1-k)(\mathbf{a} \times \mathbf{a}) \rvert$ $= \lvert k\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \sin 90^\circ \hat{\mathbf{n}} + (1-k)\mathbf{0} \rvert$ $= 9\lvert k \rvert$	
	It is the area of a parallelogram with sides OA and OC.	
(iii)	Area of triangle OAC = 3 x area of triangle OAB $\frac{1}{2}\lvert \mathbf{a} \times \mathbf{c} \rvert = \frac{3}{2}\lvert \mathbf{a} \times \mathbf{b} \rvert$ $9\lvert k \rvert = 3\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \sin 90^\circ$ $= 27$ $\lvert k \rvert = 3$ $k = \pm 3$.	
(iv)	Length of projection of OC onto $OA = 12$ $\frac{\lvert \mathbf{c} \bullet \mathbf{a} \rvert}{\lvert \mathbf{a} \rvert} = 12$ $\lvert \mathbf{c} \bullet \mathbf{a} \rvert = 12\lvert \mathbf{a} \rvert$ $= 36$	
	When $k = 3$, $\mathbf{c} = 3\mathbf{b} - 2\mathbf{a}$ $\lvert \mathbf{c} \bullet \mathbf{a} \rvert = \lvert (3\mathbf{b} - 2\mathbf{a}) \cdot \mathbf{a} \rvert$ $= \lvert 3(\mathbf{b} \cdot \mathbf{a}) - 2\mathbf{a} \cdot \mathbf{a} \rvert$ $= \lvert 3(0) - 2(3)^2 \rvert$ $= 18$	
	When $k = -3$, $\mathbf{c} = -3\mathbf{b} + 4\mathbf{a}$ $\lvert \mathbf{c} \bullet \mathbf{a} \rvert = \lvert (-3\mathbf{b} + 4\mathbf{a}) \cdot \mathbf{a} \rvert$ $= \lvert -3(\mathbf{b} \cdot \mathbf{a}) + 4\mathbf{a} \cdot \mathbf{a} \rvert$ $= \lvert -3(0) + 4(3)^2 \rvert$ $= 36$	
	$\therefore \mathbf{c} = -3\mathbf{b} + 4\mathbf{a}$	