Q1

Question

The equations of planes, P_1, P_2 are

$$P_1: \quad r = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \beta, \lambda \epsilon \mathrm{R} \qquad P_2 \quad r \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1$$

Find the coordinates of the foot of perpendicular from the point $A(-3, 10, 3)$ to the plane P_2 and show that the point $B(2, 10, -2)$ is the reflection of point A in P_2.

The planes $P_1$and P_2 meet in a line L. Find a vector equation in line L.

Plane P_3 is the reflection of P_1 in P_2. Using the results above, find a vector perpendicular to P_3. hence, find, in scalar form, the equation of P_3.

Ans: $F, \left(-\frac{1}{2}, 10, \frac{1}{2}\right), L = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix}, r. \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} = -4$

answer

8	Let F be the foot of perpendicular from A to the plane P_2. Equation of line AF is given by $r = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \ s \in \mathbb{R}$ Since F lies on the line, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + s \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, for some $s \in \mathbb{R}$. Since F also lies on plane P_2, $\begin{pmatrix} -3-s \\ 10 \\ 3+s \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1 \quad \Rightarrow (3+s)+(3+s) = 1 \quad \Rightarrow s = -\frac{5}{2}$ Therefore, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} - \frac{5}{2} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 10 \\ \frac{1}{2} \end{pmatrix}$ Coordinates of foot of perpendicular $= \left(-\frac{1}{2}, 10, \frac{1}{2}\right)$. By ratio theorem, $\overrightarrow{OF} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$

$$\Rightarrow \overline{OB} = 2\overline{OF} - \overline{OA} = 2\begin{pmatrix} -\frac{1}{2} \\ 10 \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix}$$

Hence $(2,10,-2)$ is the reflection of $(-3,10,3)$ in P_2.

When the planes intersect,

$$\left(\begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 1$$

$\Rightarrow 6 + 5\lambda = 1$

$\Rightarrow \lambda = -1$

Subst $\lambda = -1$,

$$r = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} + \beta\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$\therefore$ equation of L is

$$r = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

Alternative:

$$\begin{pmatrix} -2 \\ 12 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix}$$

$$P_1: r \cdot \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 5 \\ -14 \end{pmatrix} = -19$$

$P_1: 9x+5y-14z = -19$

$P_2: -x+z = 1$

By GC, $x=-1+z$

$y=-2+z$

$$L: r = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$A(-3,10,3)$ is on P_1, $B(2,10,-2)$ lies on plane P_3.

Line L also lies on P_3.

Therefore the vector $\begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \\ -2 \end{pmatrix}$ is parallel to P_3.

Hence a vector $\perp$ to P_3 is given by $\begin{pmatrix} 3 \\ 12 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix}$.

Equation of plane P_3 is given by:

$$r \cdot \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} \Rightarrow r \cdot \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} = -4$$

Q2

Question

Planes P_1, P_2 and P_3 have equations

$$-2x + z = 4$$

$$2x + y - 2z = 6$$

$$-6x + 4y + \lambda z = \mu$$

Respectively, where λ and μ are constants.

(i) Find a vector parallel to both P_1 and P_2.
Given that the point with coordinates $(-5, \alpha, \beta)$, lies on P_1 and P_2, find α and β.
Hence find a vector equation of the line of intersection of P_1 and P_2.

(ii) Given that P_1, P_2 and P_3 form a triangular prism, what can be said about the values of λ and μ?

Ans: (i) $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\alpha = 4, \beta = -6$, $r = \begin{pmatrix} -5 \\ 4 \\ -6 \end{pmatrix} + \delta\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ (ii) $\lambda = -1$, $\mu \neq 52$

Answer

5(i)	The vector parallel to both p_1 and p_2 $\begin{pmatrix}-2\\0\\1\end{pmatrix}\times\begin{pmatrix}2\\1\\-2\end{pmatrix}=\begin{pmatrix}-1\\-2\\-2\end{pmatrix}=\begin{pmatrix}1\\2\\2\end{pmatrix}$

5(ii)	$\begin{pmatrix}-5\\\alpha\\\beta\end{pmatrix}\cdot\begin{pmatrix}-2\\0\\1\end{pmatrix}=4$ $\quad$ $\begin{pmatrix}-5\\\alpha\\\beta\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}=6$ $10+\beta=4$ $\quad$ $-10+\alpha-2\beta=6$ $\beta=-6$ $\quad$ $\alpha=6+10+2(-6)=4$ The vector equation of the line in which p_1 and p_2 intersect is $r=\begin{pmatrix}-5\\4\\-6\end{pmatrix}+\delta\begin{pmatrix}1\\2\\2\end{pmatrix}$, $\delta\in\mathbb{R}$ Vector equation of the line in which p_1 and p_2 intersect must be perpendicular to the normal of p_3 $\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}-6\\4\\\lambda\end{pmatrix}=0$ $-6+8+2\lambda=0$ $\lambda=-1$ The point (-5, 4, -6) must not be on p_3 $\mu\neq\begin{pmatrix}-5\\4\\-6\end{pmatrix}\cdot\begin{pmatrix}-6\\4\\1\end{pmatrix}=52$

Q3

Question

The planes P_1 and P_2 have equations $r.\begin{pmatrix}-1\\-2\\2\end{pmatrix}=-1$ and $r.\begin{pmatrix}-7\\4\\4\end{pmatrix}=1$ respectively.

(i) Find the acute angle between P_1 and P_2.

(ii) The point $A(2\,,\alpha\,,3)$ is equidistant from the planes P_1 and P_2. Calculate the two possible values of α

(iii) Find the position vector of the foot of perpendicular from $B(0,1,2)$ to the plane P_1. Hence find the Cartesian equation of the plane P_3 such that P_3 is parallel to P_1 and point B is equidistant from planes P_1 and P_3

Ans: (i) 74.97° (ii) $\alpha=\frac{9}{5}$ or 6 (iii) position vector $\overline{OM}=\frac{1}{3}\begin{pmatrix}1\\5\\4\end{pmatrix}$, equation of plane $x-2y+2z=5$

Answer

8(i)	The acute angle between p_1 and p_2

$$=\cos^{-1}\left|\frac{\begin{pmatrix}-7\\4\\4\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}}{\sqrt{7^2+4^2+4^2}\sqrt{1^2+2^2+2^2}}\right|$$

$$=74.97^\circ=75.0^\circ$$

8(ii)

$\begin{pmatrix}1\\0\\0\end{pmatrix}$ is a point on p_1.

$\begin{pmatrix}1\\2\\0\end{pmatrix}$ is a point on p_2.

Let point D be $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

$$\overline{AD}=\overline{OD}-\overline{OA}$$

$$=\begin{pmatrix}1\\0\\0\end{pmatrix}-\begin{pmatrix}2\\\alpha\\3\end{pmatrix}=\begin{pmatrix}-1\\-\alpha\\-3\end{pmatrix}$$

Perpendicular distance from A to p_1

$$\left|\frac{\overline{AD}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}}{\sqrt{1^2+2^2+2^2}}\right|$$

$$=\left|\frac{\begin{pmatrix}-1\\-\alpha\\-3\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}}{\sqrt{1^2+2^2+2^2}}\right|$$

$$=\left|\frac{1+2\alpha-6}{\sqrt{9}}\right|$$

$$=\left|\frac{-5+2\alpha}{3}\right|$$

Let point C be $\begin{pmatrix}1\\2\\0\end{pmatrix}$.

$$\overline{AC}=\overline{OC}-\overline{OA}$$

$$=\begin{pmatrix}1\\2\\0\end{pmatrix}-\begin{pmatrix}2\\\alpha\\3\end{pmatrix}=\begin{pmatrix}-1\\2-\alpha\\-3\end{pmatrix}$$

Perpendicular distance from A to p_2

$$\left|\frac{\overline{AC}\cdot\begin{pmatrix}-7\\4\\4\end{pmatrix}}{\sqrt{7^2+4^2+4^2}}\right|$$

$$=\left|\frac{\begin{pmatrix}-1\\2-\alpha\\-3\end{pmatrix}\cdot\begin{pmatrix}-7\\4\\4\end{pmatrix}}{\sqrt{7^2+4^2+4^2}}\right|$$

$$=\left|\frac{7+8-4\alpha-12}{\sqrt{81}}\right|$$

$$=\left|\frac{3-4\alpha}{9}\right|$$

$$\therefore\left|\frac{-5+2\alpha}{3}\right|=\left|\frac{3-4\alpha}{9}\right|$$

$$\frac{-5+2\alpha}{3}=\frac{3-4\alpha}{9}$$

$$-15+6\alpha=3-4\alpha$$

$$10\alpha=18$$

$$\alpha=\frac{18}{10}=\frac{9}{5}=1\frac{4}{5}$$

or

$$\frac{-5+2\alpha}{3}=-\frac{3-4\alpha}{9}$$

$$-15+6\alpha=-3+4\alpha$$

$$2\alpha=12$$

$$\alpha=\frac{12}{2}=6$$

8(iii)

Let M be the position vector of the foot of perpendicular from B to p_1.

Equation of line segment BM

$$r=\begin{pmatrix}0\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\-2\\2\end{pmatrix}$$

When line segment BM intersects p_1.

$$\begin{pmatrix}-\lambda\\1-2\lambda\\2+2\lambda\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}=-1$$

$$\lambda-2+4\lambda+4+4\lambda=-1$$

$$\lambda=-\frac{1}{3}$$

$$\therefore\overline{OM}=\frac{1}{3}\begin{pmatrix}1\\5\\4\end{pmatrix}$$

Let point M' be a point in plane p_3 such that $\overline{MB}=\overline{BM'}$.

$$\overline{MB}=\overline{BM'}$$

$$\overline{OM'}=2\overline{OB}-\overline{OM}$$

$$\overline{OM'}=2\begin{pmatrix}0\\1\\2\end{pmatrix}-\frac{1}{3}\begin{pmatrix}1\\5\\4\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-1\\1\\8\end{pmatrix}$$

Equation of plane p_3

$$r.\begin{pmatrix}-1\\-2\\2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}-1\\1\\8\end{pmatrix}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix}=5$$

$$-x-2y+2z=5$$

Q4

Question

The line l_1 passes through the point $(1, -1, 1)$ and is parallel to the vector $\mathbf{2i - 5j + 3k}$.

The line l_2 has equation $x - 1 = \frac{2y+4}{4} = -\frac{z}{2}$

Given that the plane Π_1 contains l_1 and is parallel to l_2

(i) Find the Cartesian equation of the plane Π_1

(ii) Find the shortest distance between Π_1 and l_2, leaving your answer in exact form.

The plane Π_2 has equation $\mathbf{r}.(\mathbf{3i - 2k}) = \mathbf{1}$

(iii) Find the acute angle between the planes Π_1 and Π_2

(iv) Determine the geometrical relationship between Π_2 and l_1, showing your working clearly.

(v) Hence what can be said about the values of a and b such that there is no solution for the following system of linear equations?

$$4x + 7y + 9z = 6$$
$$3x - 2z = 1$$
$$2x + y - az = b$$

Ans: (i) equation of plane $4x + 7y + 9z = 6$ (ii) $\frac{16}{\sqrt{146}}$ (iii) $82.1°$ (iv) Π_2 contains l_1 (v) $a = -\frac{1}{3}, b \neq \frac{4}{3}$

Answer

9(i)	The vector equation of l_2 is $\mathbf{r} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ The normal vector of $\Pi_1 = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix}$ Scalar Product form of equation of Π_1 is $\mathbf{r} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} = 6$ Cartesian equation of the plane Π_1 is $4x + 7y + 9z = 6$
(ii)	Let d be the shortest distance. Then $d = \frac{\left| \overline{AB} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} \right|}{\sqrt{9^2 + 7^2 + 4^2}}$, where $A = (1, -1, 1)$ and $B(1, -2, 0)$

	$= \frac{\left\| \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} \right\|}{\sqrt{146}}$ $= \frac{16}{\sqrt{146}}$
(iii)	Let θ be the angle between the two planes. $\cos\theta = \frac{\left\| \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 7 \\ 9 \end{pmatrix} \right\|}{\sqrt{13}\sqrt{146}} = \frac{6}{\sqrt{13}\sqrt{146}}$ $\theta = 82.1°$
(iv)	Note that (1, -1, 1) lies on Π_2 since $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = 1$ Also, note that l_1 is parallel to Π_2 since $\begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = 0$ So Π_2 contains l_1.
(v)	Since both Π_1 and Π_2 contain l_1, l_1 is parallel to the plane $2x + y - az = b$

	So $\begin{pmatrix} 2 \\ 1 \\ -a \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} = 0$ $\Rightarrow a = -\frac{1}{3}$ Note that (1, -1, 1) does not lie on plane $2x + y - az = b$ So $2(1) - 1 - a(1) \neq b \Rightarrow b \neq \frac{4}{3}$.

Q5

Question

The planes p_1 and p_2 have Cartesian equations $2x + y - 3z = -5$ and

$5x - 7y + 2z = -3$ respectively and meet in a line l.

(i) Find the acute angle between p_1 and p_2

(ii) Find a vector equation of l

The line l_1 passes through the point A with position vector $-\mathbf{2k}$, is parallel to p_1 and is perpendicular to l.

(iii) Explain briefly why l and l_1 are skew lines.

(iv) Find a vector equation of l_1

The plane p_3 contains l_1 and is parallel to p_1

(v) Find the perpendicular distance between p_1 and p_3

(vi) Hence state the perpendicular distance between l and l_1.

Ans: (i) 84.8° (ii) $r = \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ (iv) $r = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}$ (v) $\frac{11}{\sqrt{14}}$ (vi) $\frac{11}{\sqrt{14}}$

Answer

2(i)

$$\cos\theta = \frac{\left|\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}5\\-7\\2\end{pmatrix}\right|}{\sqrt{14}\sqrt{78}} = \frac{|10-7-6|}{\sqrt{14}\sqrt{78}} = \frac{3}{\sqrt{14}\sqrt{78}}$$

$\Rightarrow \theta = 84.8°$

2(ii)

From GC, $\ell : r = \begin{pmatrix}-2\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix},\ \lambda \in \mathbb{R}$

2(iii)

Since ℓ and ℓ_1 are perpendicular $\Rightarrow \ell$ is not parallel to ℓ_1.

Note that ℓ_1 is parallel to p_1 and A is not on p_1 $\therefore$ ℓ and ℓ_1 do not intersect.

$\Rightarrow \ell_1$ is not on p_1.

Hence, ℓ and ℓ_1 are skew lines.

2(iv)

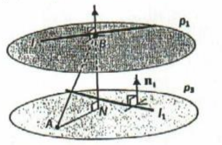
Direction vector of $\ell_1 = \begin{pmatrix}2\\1\\-3\end{pmatrix}\times\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}4\\-5\\1\end{pmatrix}$

$\therefore \ell_1 : r = \begin{pmatrix}0\\0\\-2\end{pmatrix} + \mu\begin{pmatrix}4\\-5\\1\end{pmatrix},\ \mu \in \mathbb{R}$

2(v)

Method 1 (Compute length of project vector)

Choose $A = (0,0,-2)$ on p_3 and $B = (-2,-1,0)$ on p_1.



Required perpendicular distance is

$$\left|\overrightarrow{AB}\cdot\hat{n}\right| = \frac{1}{\sqrt{14}}\left|\begin{pmatrix}-2\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-3\end{pmatrix}\right| = \frac{1}{\sqrt{14}}|-4-1-6| = \frac{11}{\sqrt{14}}.$$

Method 2 (Use of foot of perpendicular)

Choose $A = (0,0,-2)$ on p_3.

Obtain equation of line through A and perpendicular to $p_1 \Rightarrow r = \begin{pmatrix}0\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-3\end{pmatrix}$

Find foot of perpendicular N through intersection of line and p_1

$$\Rightarrow \left(\begin{pmatrix}0\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-3\end{pmatrix}\right)\cdot\begin{pmatrix}2\\1\\-3\end{pmatrix} = -5 \Rightarrow \lambda = \frac{-11}{14}$$

Thus $\overrightarrow{NA} = \overrightarrow{ON} - \overrightarrow{OA} = \lambda\begin{pmatrix}2\\1\\-3\end{pmatrix} = \frac{-11}{14}\begin{pmatrix}2\\1\\-3\end{pmatrix}$ and required perpendicular distance is

$$NA = \frac{11}{14}\left|\begin{pmatrix}2\\1\\-3\end{pmatrix}\right| = \frac{11\sqrt{14}}{14}.$$

Perpendicular distance between ℓ and ℓ_1 equals to the perpendicular distance between the two planes $= \frac{11}{\sqrt{14}}$.

(Refer to same diagram above)