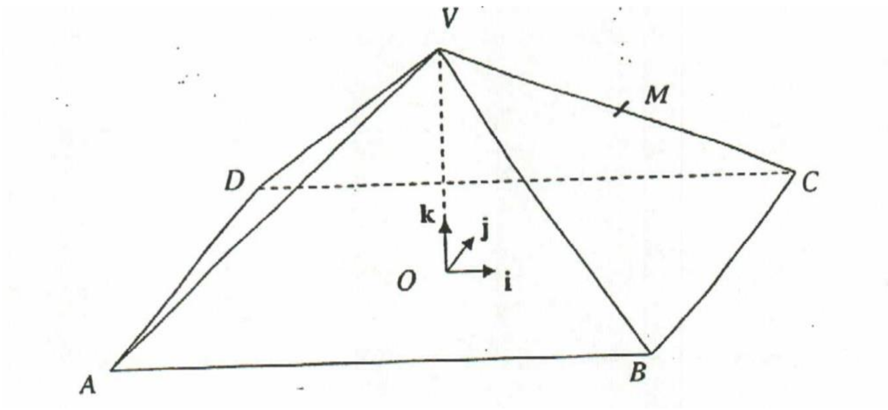
Q1

Question



In the diagram, *O* is centre of the rectangular base *ABCD* of a right pyramid with vertex *V*. Perpendicular unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are parallel to AB, BC, OV respectively. The length of AB, BC, OV are $12, 6$, and 6 units respectively. The point M is the mid-point of CV and the point O is taken as the origin for position vectors.

(i) Show that the equation of the line AM may be expressed as $\boldsymbol{r} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$, where t is a parameter.

(ii) Find the perpendicular distance from *B* to the line *AM*.

(iii) Find the acute angle between the line DV and the plane AMB.

The plane Π has equation $\boldsymbol{r}.\begin{pmatrix} -1 \\ 4 \\ a \end{pmatrix} = 4$

(iv) Given that the three planes AMB, AMD and Π have no point in common, find the value of a.

Ans: (ii) 6.18 (iii) $47.7°$ (iv) $a = -3$

Answer

4(i)

$\vec{OA} = -6\underline{i} - 3\underline{j} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix}, \vec{OC} = -\vec{OA} = \begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix}, \vec{OV} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$

$\therefore \vec{OM} = \frac{1}{2}\left[\begin{pmatrix} 6 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}\right] = \begin{pmatrix} 3 \\ \frac{3}{2} \\ 3 \end{pmatrix}$

$\therefore \vec{AM} = \begin{pmatrix} 3 \\ \frac{3}{2} \\ 3 \end{pmatrix} - \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 9 \\ \frac{9}{2} \\ 3 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$

Hence, equation of the line *AM* is $\mathbf{r} = \begin{pmatrix} -6 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}, t \in \mathbb{R}$

(ii)	$\overrightarrow{AB}=\begin{pmatrix}12\\0\\0\end{pmatrix}$ Length of projection of $\overrightarrow{AB}$ onto the line AM, $AN=\dfrac{\left\lvert\overrightarrow{AB}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}\right\rvert}{\sqrt{6^2+3^2+2^2}}$ $=\dfrac{\left\lvert\begin{pmatrix}12\\0\\0\end{pmatrix}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}\right\rvert}{7}=\dfrac{72}{7}$ Perpendicular distance from P to the line AM $=\sqrt{(AB)^2-(AN)^2}$ $=\sqrt{12^2-\left(\dfrac{72}{7}\right)^2}=6.18$ (3 s.f.) Alternative Method Perpendicular distance from P to the line AM $=\dfrac{\left\lvert\overrightarrow{AB}\times\begin{pmatrix}6\\3\\2\end{pmatrix}\right\rvert}{\sqrt{6^2+3^2+2^2}}=\dfrac{\left\lvert\begin{pmatrix}12\\0\\0\end{pmatrix}\times\begin{pmatrix}6\\3\\2\end{pmatrix}\right\rvert}{\sqrt{49}}$ $=\dfrac{12\left\lvert\begin{pmatrix}0\\-2\\3\end{pmatrix}\right\rvert}{7}=\dfrac{12\sqrt{4+9}}{7}$ $=6.18$ (3 s.f.)
(iii)	A normal vector to the plane AMB $=\begin{pmatrix}12\\0\\0\end{pmatrix}\times\begin{pmatrix}6\\3\\2\end{pmatrix}=12\begin{pmatrix}0\\-2\\3\end{pmatrix}$ $\overrightarrow{DV}=\begin{pmatrix}0\\0\\6\end{pmatrix}-\begin{pmatrix}-6\\3\\0\end{pmatrix}=\begin{pmatrix}6\\-3\\6\end{pmatrix}=3\begin{pmatrix}2\\-1\\2\end{pmatrix}$ Let θ be that angle betw. the line DV and plane AMB.
	$\sin\theta=\dfrac{\left\lvert\begin{pmatrix}2\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}0\\-2\\3\end{pmatrix}\right\rvert}{\sqrt{4+1+4}\sqrt{4+9}}$ $=\dfrac{2+6}{3\sqrt{13}}$ $\theta=47.7^\circ$ (1 d.p.)
(iv)	If the 3 planes AMB, AMD and Π do not have a common point, the line AM is parallel to Π but does not lie in Π. $\therefore \begin{pmatrix}6\\3\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\a\end{pmatrix}=0$ $\Rightarrow -6+12+2a=0$ $\Rightarrow a=-3$ Note that $\begin{pmatrix}-6\\-3\\0\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\a\end{pmatrix}=6-12\neq 4$. Therefore point A does not lie in Π. Hence the line AM does not lie in Π.

Q2

Question

The lines l_1 and l_2 have equations

$$l_1: \boldsymbol{r}=\begin{pmatrix}1\\4\\1\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\1\end{pmatrix},\ \lambda\in R \quad\text{and}\quad l_2: \boldsymbol{r}=\begin{pmatrix}5\\2\\-1\end{pmatrix}+\mu\begin{pmatrix}3\\1\\-1\end{pmatrix}, \mu\in R$$

(i) Determine whether the lines l_1 and l_2 are perpendicular

(ii) Lines l_1 and l_2 intersect at the point P. Find the coordinates of P.

The points A and B have position vectors $\begin{pmatrix}1\\4\\1\end{pmatrix}$ and $\begin{pmatrix}5\\2\\-1\end{pmatrix}$ respectively. Line L is a perpendicular bisector of the line segment AB. The plane π contains all such L.

(iii) Find an equation of π in the form of $\mathbf{r}.\mathbf{n}=D$

(iv) Verify that plane π contains point P.

(v) Using your answers in parts (i) and (iv), give an geometrical interpretation of $|\overrightarrow{PA} \times \overrightarrow{PB}|$

Ans: (i) not perpendicular (ii) $(2,1,0)$ (iii) $\mathbf{r} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -3$ (v) area of the rhombus in which PA and PB are adjacent sides.

Answer

Qn	Solution
2	**Vectors**
(i)	$\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = -1 \neq 0$ Therefore, l_1 and l_2 are not perpendicular.
(ii)	$\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ $1-\lambda = 5+3\mu \Rightarrow -\lambda - 3\mu = 4$ --- (1) $4+3\lambda = 2+\mu \Rightarrow 3\lambda - \mu = -2$ --- (2) $1+\lambda = -1-\mu \Rightarrow \lambda + \mu = -2$ --- (3) Using GC, $\lambda = -1, \mu = -1$ $\Rightarrow \overrightarrow{OP} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ $\therefore$ coordinates of P are $(2,1,0)$
(iii)	Since all the lines in π are perpendicular to AB, $\mathbf{n}$, the normal vector of π, is parallel to $\overrightarrow{AB}$. $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$ $\therefore \mathbf{n} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ π passes through M, the midpoint of A and B. $\overrightarrow{OM} = \frac{1}{2}\left[\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$ $\pi : \mathbf{r} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -6+3 = -3$ $\therefore \pi : \mathbf{r} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -3$
(iv)	$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = -4+1 = -3$ Hence, plane π contains point P.

(v)	Plane π contains point $P \Rightarrow AP = BP$. Moreover, l_1 and l_2 are not perpendicular. Hence, $\lvert\overrightarrow{PA} \times \overrightarrow{PB}\rvert$ gives the area of the **rhombus** in which PA and PB are adjacent sides. Do not accept: Area of parallelogram Area of square Area of rectangle $2 \times$ area of ΔPAB

Q3

Question

The lines l_1 and l_2 have equations

$l_1: \boldsymbol{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$, $\lambda \in R$ and $l_2: \boldsymbol{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \mu \in R$

Respectively.

(i) Show that l_1 and l_2 are skew lines.

(ii) Find the acute angle between the lines l_1 and l_2.

Ans: (ii) 33.6°

Answer

1i	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ Since $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \neq m \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ for $m \in \mathbb{R}$, the lines are not parallel. $-2\lambda = 1 + \mu$ --- (1) $\lambda = 1 - 2\mu$ ----- (2) $1 - \lambda = \mu$ ------- (3) Solving (1) and (2): $\lambda = -1$, $\mu = 1$ when $\lambda = 1$, $\mu = 0$, (1): LHS $= -2 \neq 1 =$ RHS Therefore the lines are skewed.
1ii	angle $= \cos^{-1} \dfrac{\left\| \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|}{\sqrt{6}\sqrt{6}}$ $= 33.6°$

Q4

Question

The lines l_1 and l_2 meet at the point P. The line l_3 is coplanar with l_1 and l_2 and is perpendicular to l_1. Given that l_1 and l_2 are parallel to the vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, show that l_3 is parallel to the vector $-\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$.

The equations of l_1 and l_2 are now known to be $\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + l \begin{pmatrix} 11 \\ 10 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + s \begin{pmatrix} 17 \\ 3 \\ 4 \end{pmatrix}$ respectively, where s and l are real parameters. Find the equation of the line l_3, given that l_3 also passes through P.

The line l_4 has equation $r = \begin{pmatrix} 10 \\ -3 \\ 1 \end{pmatrix} + u \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$, where u is a real parameter. Determine if l_3 and l_4 are skew or intersecting.

The line l_5 is perpendicular to both l_3 and l_4. Find the acute angle between l_5 and the plane containing l_1 and l_2 .

Ans: $l_3 := \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ -7 \\ 2 \end{pmatrix}$, acute angle= 83.9°

Answer

9 A direction vector parallel to l_3 is given by $\alpha\mathbf{b}-\beta\mathbf{a}$

Since l_3 is perpendicular to l_1, $\mathbf{a}\cdot(\alpha\mathbf{b}-\beta\mathbf{a})=0 \Rightarrow \beta=\dfrac{\alpha\mathbf{a}.\mathbf{b}}{|\mathbf{a}|^2}$

Therefore, l_3 is parallel to $\alpha\mathbf{b}-\dfrac{\alpha\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$, i.e. $\mathbf{b}-\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}$

Direction vector of l_3 is given by $\mathbf{b}-\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a}=\begin{pmatrix}17\\3\\4\end{pmatrix}-\dfrac{\begin{pmatrix}11\\10\\2\end{pmatrix}.\begin{pmatrix}17\\3\\4\end{pmatrix}}{\left|\begin{pmatrix}11\\10\\2\end{pmatrix}\right|^2}\begin{pmatrix}11\\10\\2\end{pmatrix}=\begin{pmatrix}6\\-7\\2\end{pmatrix}$

Equation of l_3 is $\mathbf{r}=\begin{pmatrix}3\\-5\\2\end{pmatrix}+\lambda\begin{pmatrix}6\\-7\\2\end{pmatrix}$, where $\lambda\in\mathbb{R}$.

Clearly, the direction vectors $\begin{pmatrix}6\\-7\\2\end{pmatrix}$ and $\begin{pmatrix}-3\\4\\-1\end{pmatrix}$ are non-parallel. Hence the two lines are not parallel.

Equating $\begin{pmatrix}3+6\lambda\\-5-7\lambda\\2+2\lambda\end{pmatrix}=\begin{pmatrix}10-3\mu\\-3+4\mu\\1-\mu\end{pmatrix}$,

there are no unique values for λ and μ that satisfy the 3 equations. Therefore l_3 and l_4 are skew.

$\begin{pmatrix}6\\-7\\2\end{pmatrix}\times\begin{pmatrix}-3\\4\\-1\end{pmatrix}=\begin{pmatrix}-1\\0\\3\end{pmatrix}$

Vector normal to both l_3 and l_4 is $\begin{pmatrix}-1\\0\\3\end{pmatrix}$, which is direction vector of l_5

$\begin{pmatrix}11\\10\\2\end{pmatrix}\times\begin{pmatrix}17\\3\\4\end{pmatrix}=\begin{pmatrix}34\\-10\\-137\end{pmatrix}$

Vector normal to plane containing l_1 and l_2 is $\begin{pmatrix}-34\\10\\137\end{pmatrix}$

Let required angle be θ.

Using $\sin\theta=\dfrac{\begin{pmatrix}-1\\0\\3\end{pmatrix}.\begin{pmatrix}-34\\10\\137\end{pmatrix}}{\sqrt{10\times 20025}} \Rightarrow \theta=83.9^\circ$