Q1

Question

A curve has parametric equations $x = \frac{1}{t^2}$, $y = 2t$, where t is a non-zero parameter.

(i) Sketch the curve.

The region R is bounded by the curve and the lines $x = 4$ and $x = 16$.

(ii) Find the area of R.

(iii) Find the exact value of $\int_{\frac{1}{2}}^{\frac{1}{4}} \frac{1}{t}\, dt$. Hence find the volume of the solid generated when the region R is rotated through π radians about the x-axis.

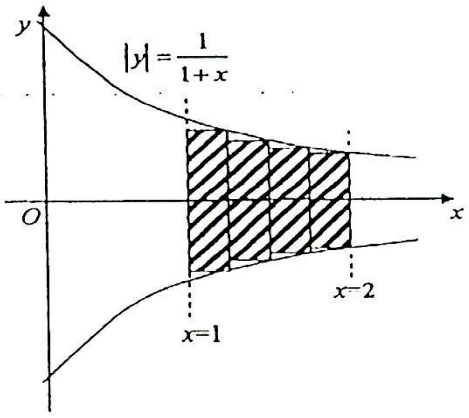
Solution

Q2

Question

The diagram shows the parts of the graph of $|y| = \frac{1}{1+x}$. The four rectangles each of equal width, as shown in diagram below, are rotated through π radians about the x-axis. Prove that the volume generated may be expressed as

$$\sum_{r=1}^{4} \frac{4\pi}{(8+r)^2}.$$ [2]



n rectangles, each of equal width under the curve between $x = 1$ and $x = 2$, are used to estimate the volume of solid obtained when the region bounded by the graph, the lines $x = 1$ and $x = 2$, is rotated through π radians about the x-axis. Find the estimated volume in the form $\sum_{r=1}^{n} f(r)$, where $f(r)$ is to be determined in terms of r and n. [2]

(i) Deduce that $\sum_{r=1}^{n} \frac{1}{(2n+r)^2} < \frac{1}{6n}$. [3]

(ii) State the value of $\sum_{r=1}^{n} \frac{n}{(2n+r)^2}$ as $n \to \infty$. [1]

Solution

Width = $\frac{1}{n}$

Rectangle number	1	2	r	n
x-coor	$1+\frac{1}{n}$	$1+\frac{2}{n}$	$1+\frac{r}{n}$	$1+\frac{n}{n}$
y-coor	$\frac{1}{1+(1+\frac{1}{n})}$	$\frac{1}{1+(1+\frac{2}{n})}$	$\frac{1}{1+(1+\frac{r}{n})}$	$\frac{1}{1+(1+\frac{n}{n})}$
Vol	$\pi\left(\frac{1}{1+\left(1+\frac{1}{n}\right)}\right)^2 \left(\frac{1}{n}\right.$	$\pi\left(\frac{1}{1+\left(1+\frac{2}{n}\right)}\right)^2 \left(\frac{1}{n}\right.$	$\pi\left(\frac{1}{1+\left(1+\frac{r}{n}\right)}\right)^2 \left(\frac{1}{n}\right.$	$\pi\left(\frac{1}{1+\left(1+\frac{n}{n}\right)}\right)^2 \left(\frac{1}{n}\right.$

$$Vol = \pi\sum_{1}^{n}\left(\frac{1}{1+\left(1+\frac{r}{n}\right)}\right)^2\left(\frac{1}{n}\right) = \pi\sum_{1}^{n}\frac{1}{\left(2+\frac{r}{n}\right)^2 n} = \pi\sum_{1}^{n}\frac{n}{\left(2+\frac{r}{n}\right)^2 n^2}$$

$$= \sum_{1}^{n}\frac{n\pi}{(2n+r)^2}$$

$$Vol\ under\ curve = \pi\int_{1}^{2}\frac{1}{(1+x)^2}dx = \pi\left[-\frac{1}{1+x}\right]_1^2 = \frac{\pi}{6}$$

Hence $\sum_1^n \frac{n\pi}{(2n+r)^2} < \frac{\pi}{6}$

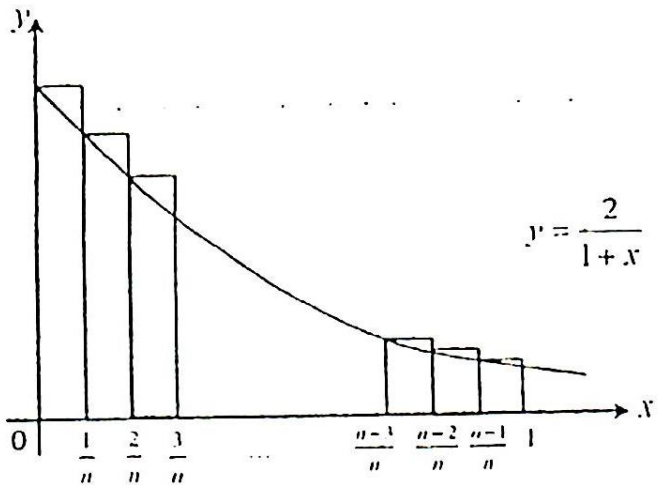
$$\sum_{1}^{n}\frac{1}{(2n+r)^2} < \frac{1}{6n}$$

ii) $S = \frac{1}{6}$

Q3

Question

The graph of $y = \frac{2}{1+x}$ for $x \geq 0$, is shown in the diagram below, Region R is bounded by the x-axis, the y-axis, the line $x = 1$ and the curve $y = \frac{2}{1+x}$. The area of region R may be approximated by the total area, A, of n rectangles, each of width $\frac{1}{n}$, as shown in the diagram.



(i) Show that

$$A = \sum_{r=0}^{n-1} \frac{2}{n+r}$$

(ii) By considering the exact area of region R, show that $\sum_{r=0}^{n-1} \frac{1}{n+r} > \ln 2$

Solution

Width = $\frac{1}{n}$

Rectangle number	1	2	r	n
x-coor	$0+\frac{0}{n}$	$0+\frac{1}{n}$	$0+\frac{r-1}{n}$	$0+\frac{n-1}{n}$
y-coor	$\frac{2}{1+0+\frac{0}{n}}$	$\frac{2}{1+0+\frac{1}{n}}$	$\frac{2}{1+0+\frac{r-1}{n}}$	$\frac{2}{1+0+\frac{n-1}{n}}$
Area	$\frac{2}{1+0+\frac{0}{n}}\left(\frac{1}{n}\right)$	$\frac{2}{1+0+\frac{1}{n}}\left(\frac{1}{n}\right)$	$\frac{2}{1+0+\frac{r-1}{n}}\left(\frac{1}{n}\right)$	$\frac{2}{1+0+\frac{n-1}{n}}\left(\frac{1}{n}\right)$

$$A = \sum_{1}^{n} \frac{2}{1+0+\frac{r-1}{n}}\left(\frac{1}{n}\right) = \sum_{1}^{n} \frac{2}{n+r-1}$$

$$= \sum_{r=0}^{n-1} \frac{2}{n+r}$$

$$Area\ exact = \int_0^1 \frac{2}{1+x}dx = [2\ln(1+x)]_0^1 = 2\ln 2$$

Hence $\sum_{r=0}^{n-1} \frac{2}{n+r} > 2\ln 2$

$$\sum_{r=0}^{n-1} \frac{1}{n+r} > \ln 2$$

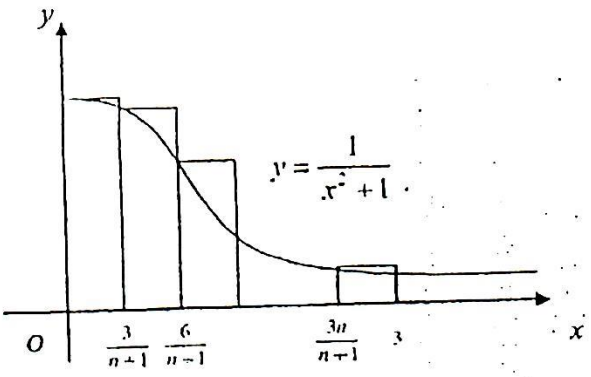
Q4

Question

11 (a) **(i)** The region R is bounded by the curves $y=\frac{4}{x^2+1}$, $y=\ln(x+1)$ and the line $x=1$ and the y-axis. Find the area of region R. **[2]**

(ii) Find the exact volume of the solid formed when R is rotated 2π radians about the y-axis. **[4]**

(b) The diagram shows part of the graph of $y=\frac{1}{x^2+1}$.



(i) By considering $n+1$ rectangles of equal width from $x=0$ to $x=3$, show that for all non-negative integers n,

$$A<\sum_{r=0}^{n}\frac{3(n+1)}{9r^2+(n+1)^2},$$

where A is the area bounded by the curve, the axes and the line $x=3$. **[3]**

(ii) Deduce $\lim_{n\to\infty}\sum_{r=0}^{n}\frac{3(n+1)}{9r^2+(n+1)^2}$ exactly. **[2]**

Solution

b)Width = $\frac{3}{n+1}$

Rectangle number	1	2	r	n+1
x-coor	$0+\frac{0}{n+1}$	$0+\frac{3}{n+1}$	$0+\frac{3(r-1)}{n+1}$	$0+\frac{3(n)}{n+1}$
y-coor	$\frac{1}{\left(\frac{0}{n+1}\right)^2+1}$	$\frac{1}{\left(\frac{3}{n+1}\right)^2+1}$	$\frac{1}{\left(\frac{3(r-1)}{n+1}\right)^2+1}$	$\frac{1}{\left(\frac{3n}{n+1}\right)^2+1}$

$$A = \sum_{1}^{n+1} \frac{1}{\left(\frac{3(r-1)}{n+1}\right)^2 + 1}\left(\frac{3}{n+1}\right) = \sum_{0}^{n} \frac{1}{\left(\frac{3r}{n+1}\right)^2 + 1}\left(\frac{3}{n+1}\right)$$

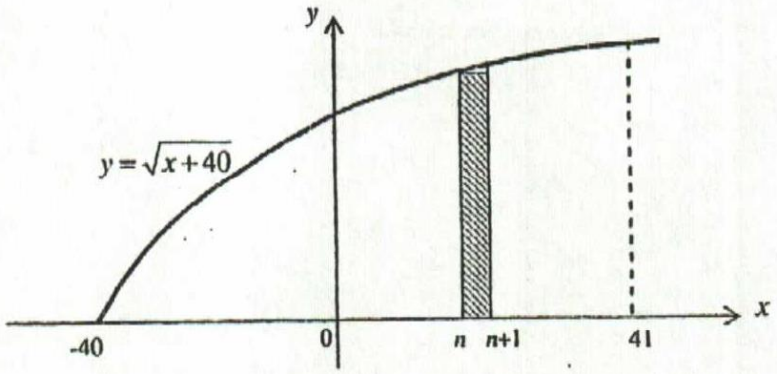
$$= \sum_{0}^{n} \frac{1}{\left(\frac{3r}{n+1}\right)^2 + 1}\left(\frac{3(n+1)}{(n+1)^2}\right) = \sum_{0}^{n} \frac{3(n+1)}{9r^2 + (n+1)^2}$$

ii) Limit = Exact area $= \tan^{-1} 3 - \tan^{-1} 0 = \tan^{-1} 3$

Q5

Question

The graph of $y = \sqrt{x+40}$ is shown in the diagram below.



By considering the shaded rectangle, show that

$$\sqrt{n+40} < \int_{n}^{n+1} \sqrt{x+40}\, dx. \qquad [1]$$

Deduce that

$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{80} < \int_{-40}^{41} \sqrt{x+40}\, dx. \qquad [2]$$

Show also that $\sqrt{n+41} > \int_{n}^{n+1} \sqrt{x+40}\, dx.$ [2]

Deduce that

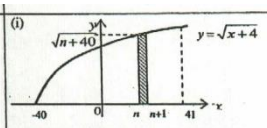
$$\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{81} > \int_{-40}^{41} \sqrt{x+40}\, dx. \qquad [1]$$

Hence, deduce the value a, where $a \in \mathbb{Z}$, that satisfies the following inequality.

$$9a < \sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{80} < 9(a+1). \qquad [2]$$

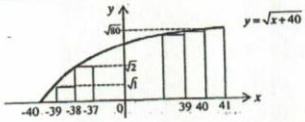
Answer

9 (i)



area of shaded rectangle < Area under the curve from x= n to x = n+1

hence $1 \times \sqrt{n+40} < \int_{n}^{n+1} \sqrt{x+40}\ dx$

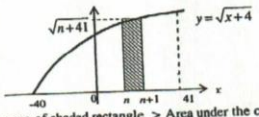


Drawing similar triangles as shown above,
Sum of area of rectangles < actual area under curve

$\sqrt{0}\times1+\sqrt{1}\times1+\sqrt{2}\times1+....+\sqrt{80}\times1 < \int_{-40}^{41} \sqrt{x+40}\ dx$

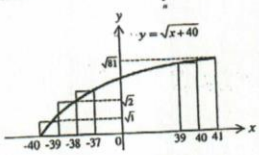
Hence $\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{80} < \int_{-40}^{41} \sqrt{x+40}\ dx$ --- (1)

Draw rectangle as shown below:



area of shaded rectangle > Area under the curve from x= n to x = n+1

hence $1 \times \sqrt{n+41} > \int_{n}^{n+1} \sqrt{x+40}\ dx$



By drawing rectangles as shown above,
Sum of area of rectangles > actual area under curve

$\sqrt{1}\times1+\sqrt{2}\times1+....+\sqrt{81}\times1 > \int_{-40}^{41} \sqrt{x+40}\ dx$

Hence $\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{81} > \int_{-40}^{41} \sqrt{x+40}\ dx$ --- (2)

$\int_{-40}^{41} \sqrt{x+40}\ dx = \left[\frac{2}{3}(x+40)^{\frac{3}{2}}\right]_{-40}^{41} \left[\frac{2}{3}(x+40)^{\frac{3}{2}}\right]_{-40}^{41} = 486$ (or use GC)

From (1) and (2),

$\int_{-40}^{41} \sqrt{x+40}\ dx - \sqrt{81} < \sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{80} < \int_{-40}^{41} \sqrt{x+40}\ dx$

$477 < \sqrt{1}+\sqrt{2}+\sqrt{3}+..+\sqrt{80} < 486$

$9(53) < \sqrt{1}+\sqrt{2}+\sqrt{3}+..+\sqrt{80} < 9(54)$

$a = 53$

Q6

Question

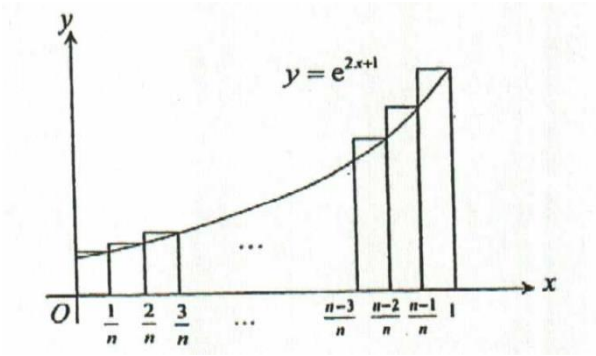
(a)

(i) Show that $(k+1)!\,k - k!\,(k-1) = k!\,(k^2+1)$.

(ii) Hence find $\sum_{k=1}^{n} k!\,(k^2+1)$

(iii) Using your answer in part (ii), find $\sum_{k=1}^{n-1}(k+1)!\,(k^2+2k+2)$.

(b) The graph of $y = e^{2x+1}$, for $0 \le x \le 1$, is shown in the diagram. Rectangles of equal width are drawn as shown in the interval between $x = 0$ and $x = 1$.



(i) Show that the total area of all the n rectangles, A, is given by $\frac{e}{n}\left(\frac{e^{\frac{2}{n}}(1-e^2)}{1-e^{\frac{2}{n}}}\right)$.

(ii) By considering the area under the curve $y = e^{2x+1}$, find the exact value of the limit of A as $n \to \infty$

(iii) Hence show that $\frac{e^{\frac{2}{n}}}{n(e^{\frac{2}{n}}-1)} > k$, where k is a constant to be found. Find the largest possible value of k.

Answer

11	Summation & Integration
ai	LHS$=(k+1)!k-k!(k-1)$ $=k!(k(k+1)-(k-1))$ $=k!(k^2+1)=$ RHS
(ii)	$\sum_{k=1}^{n}k!(k^2+1)=\sum_{k=1}^{n}[(k+1)!k-k!(k-1)]$ $=2!(1)-1!(0)$ $+3!(2)-2!(1)$ $+4!(3)-3!(2)$ $+\ldots$ $+(n-1)!(n-2)-(n-2)!(n-3)$ $+n!(n-1)-(n-1)!(n-2)$ $+(n+1)!n-n!(n-1)$ $=(n+1)!n$
(iii)	$\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)=\sum_{k-1=1}^{k-1=n-1}(k-1+1)!((k-1)^2+2(k-1)+2)$ (replace k by $k-1$) $=\sum_{k=2}^{n}k!(k^2+1)$ $=\sum_{k=1}^{n}k!(k^2+1)-1!(1^2+1)$ $=(n+1)!n-2$ Alternative Method: (Strongly not recommended) Result: $\sum_{k=1}^{n}k!(k^2+1)=(n+1)!n$ Replace k by $k+1$, $\sum_{k+1=1}^{k+1=n}(k+1)!((k+1)^2+1)=(n+1)!n$ $\Rightarrow\sum_{k=0}^{n-1}(k+1)!(k^2+2k+2)=(n+1)!n$ $\therefore\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)$ $=\sum_{k=0}^{n-1}(k+1)!(k^2+2k+2)-(0+1)!(0^2+2(0)+2)$ $=\sum_{k=1}^{n}k!(k^2+1)-2$ $=(n+1)!n-2$

(b) (i)	$A=\frac{1}{n}\left[e^{\frac{2}{n}+1}+e^{\frac{4}{n}+1}+\ldots+e^{\frac{2(n-1)}{n}+1}+e^3\right]$ $=\frac{e}{n}\left[e^{\frac{2}{n}}+e^{\frac{4}{n}}+\ldots+e^{\frac{2(n-1)}{n}}+e^2\right]$ $=\frac{e}{n}\left[\frac{e^{\frac{2}{n}}\left(1-\left(e^{\frac{2}{n}}\right)^n\right)}{1-e^{\frac{2}{n}}}\right]$ (Applying GP sum formula) $=\frac{e}{n}\left[\frac{e^{\frac{2}{n}}(1-e^2)}{1-e^{\frac{2}{n}}}\right]$ (shown)
(ii)	As $n\to\infty$, $A\to\int_0^1 e^{2x+1}\,dx=\left[\frac{1}{2}e^{2x+1}\right]_0^1$ $=\frac{1}{2}e^3-\frac{1}{2}e$ $=\frac{e}{2}(e^2-1)$
(iii)	Since the sum of the area of rectangles is an overestimate, $\frac{e}{n}\left[\frac{e^{\frac{2}{n}}(1-e^2)}{1-e^{\frac{2}{n}}}\right]>\frac{e}{2}(e^2-1)$ $\Rightarrow\frac{e}{n}\left[\frac{e^{\frac{2}{n}}(e^2-1)}{e^{\frac{2}{n}}-1}\right]>\frac{e}{2}(e^2-1)$ $\Rightarrow\frac{e^{\frac{2}{n}}}{n\left(e^{\frac{2}{n}}-1\right)}>\frac{1}{2}$ ($\div e(e^2-1)$ on both sides since $e(e^2-1)>0$) $\therefore k=\frac{1}{2}$