**Integration and its Application (I)**

Q1

**Question**

Find $S = \int x^2 \ln x \, dx$

$$u = lnx, \frac{dv}{dx} = x^2, \frac{du}{dx} = \frac{1}{x}, v = \frac{x^3}{3}$$
$$S = \frac{x^3}{3}\ln x - \int \frac{x^2}{3} dx = \frac{x^3}{3} lnx - \frac{x^3}{9} + c$$

Q2

Find $S = \int e^x \tan^{-1} e^x \, dx$

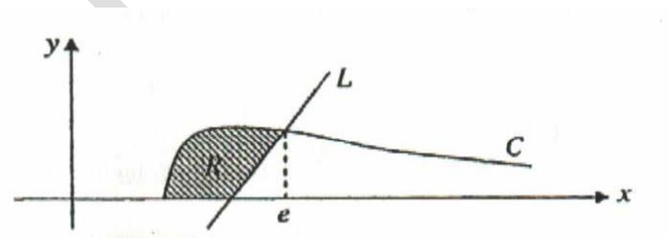
$$Let\ u = \tan^{-1} e^x, \frac{dv}{dx} = e^x, \frac{du}{dx} = \frac{e^x}{e^{2x}+1}, v = e^x$$
$$S = e^x \tan^{-1} e^x - \int e^x \frac{e^x}{e^{2x}+1} dx$$
$$= e^x \tan^{-1} e^x - \int \frac{e^{2x}}{e^{2x}+1} dx$$
$$\int \frac{e^{2x}}{e^{2x}+1} dx = \int \frac{\frac{1}{2}\frac{dk}{dx}}{k+1} dx = \frac{1}{2}\ln|k+1| = \frac{1}{2}\ln|e^{2x}+1|$$
$$S = e^x \tan^{-1} e^x - \frac{1}{2}\ln|e^{2x}+1| + c$$

Q3

**Question**

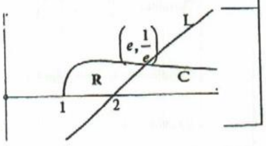
a) $\int \frac{6+2x}{\sqrt{1-4x-x^2}} dx$

b) The diagram below shows the region $R$ bounded by the curve $C$ with equation $y = \frac{\sqrt{\ln x}}{x}, x \geq 1$, the $x$-axis and the line $L$ with equation $y = \frac{1}{e(e-2)}(x-2)$. Find the exact volume of the solid of revolution when $R$ is rotated completely about the $x$-axis




Answer

| | |
|---|---|
| 3a | $\int \frac{6+2x}{\sqrt{1-4x-x^2}}\,dx = \int \frac{2-(-4-2x)}{\sqrt{1-4x-x^2}}\,dx$ <br> $= \int \frac{2}{\sqrt{1-4x-x^2}}\,dx - \int \frac{(-4-2x)}{\sqrt{1-4x-x^2}}\,dx$ <br> $= \int \frac{2}{\sqrt{5-(x+2)^2}}\,dx - \int \frac{-4-2x}{\sqrt{1-4x-x^2}}\,dx$ <br> $= 2\sin^{-1}\left(\frac{x+2}{\sqrt{5}}\right) - 2\sqrt{1-4x-x^2} + c$ |
| 3b | Point of intersection: $\left(e, \frac{1}{e}\right)$ <br> Volume <br> $= \pi\int_1^e \left(\frac{\sqrt{\ln x}}{x}\right)^2 dx - \frac{\pi}{3}\left(\frac{1}{e}\right)^2 (e-2)$ |




| | |
|---|---|
| | $= \pi\int_1^e \frac{\ln x}{x^2}\,dx - \frac{\pi(e-2)}{3e^2}$ <br> $= \pi\left[(\ln x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\frac{1}{x}\,dx\right]_1^e - \frac{\pi(e-2)}{3e^2}$ <br> $= \pi\left[\left(-\frac{\ln x}{x}\right) - \frac{1}{x}\right]_1^e - \frac{\pi(e-2)}{3e^2}$ <br> $= \pi\left[1 - \frac{2}{e}\right] - \frac{\pi(e-2)}{3e^2}$ <br> $= \pi - \frac{2\pi}{e} - \frac{\pi}{3e} + \frac{2\pi}{3e^2}$ <br> $= \pi\left(1 - \frac{7}{3e} + \frac{2}{3e^2}\right)$ |

Q4

**Question**

(i) Using the substitution $y = \ln x$, show that $\int (\ln x)^2\,dx = \int y^2 e^y\,dy$. Hence show that $\int (\ln x)^2\,dx = x\,\{(\ln x)^2 - 2(\ln x) + 2\} + c$.

(ii) It is given that $f(x) = \begin{cases} \ln x, & for\ \ 1 \le x < e \\ 0, & for\ \ e \le x \le 3 \end{cases}$

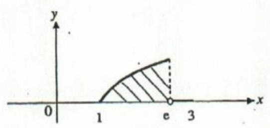
And that $f(x+2) = f(x)$ for all real values of $x$.

(a) The region bounded by the curve $y = f(x)$, the $x$-axis and the line $x = e$ is rotated through $2\pi$ radians about the $x$-axis. Find the exact value of the volume obtained.

(b) Sketch the graph of $y = f(x)$ for $-3 \le x \le 4$.

Hence find $\int_{-3}^{4} f(x)\,dx$.

Answer

| | |
|---|---|
| 10 (ii) (a) |  |



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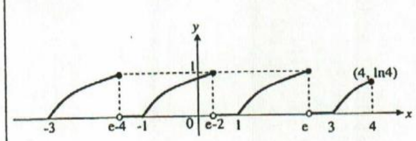
$$\text{Volume} = \pi\int_1^e (\ln x)^2\, dx$$
$$= \pi\left[x\{(\ln x)^2 - 2(\ln x) + 2\}\right]_1^e$$
$$= \pi\left[\left(e(1)^2 - 2e + 2e\right) - (2)\right]$$
$$= \pi(e-2)\ \text{units}^3$$

**10 (ii) (b)**




Method 1:

$$\int_{-3}^{4} f(x)\,dx = 3\int_1^e \ln x\,dx + \int_1^2 \ln x\,dx$$
$$= 3(1) + 0.3863$$
$$= 3.3863$$
$$= 3.39$$

Method 2:

$$\int_{-3}^{4} f(x)\,dx = 3\int_1^e \ln x\,dx + \int_3^4 \ln(x-2)\,dx$$
$$= 3(1) + 0.3863$$
$$= 3.3863$$
$$= 3.39\ (3\text{s.f})$$

Q5

**Question**

Using substitution $x = a\sin\theta$, where $a$ is a positive constant, show that

$$\int_{\frac{a}{2}}^{a} \sqrt{a^2 - x^2}\, dx = \frac{a^2}{24}(4\pi - 3\sqrt{3})$$

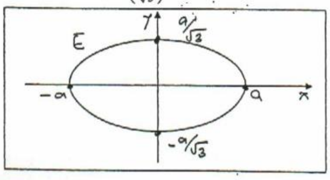
An ellipse $E$ has the equation $x^2 + 3y^2 = a^2$

i) Sketch $E$, showing clearly the coordinates of any intersections with the axes.

ii) $R$ is the region enclosed by $E$, the line $y = x$ and the positive $x$-axis. Find the exact area of $R$ in the form $k\pi a^2$.

iii) For $x > 0$, $S$ is the region enclosed by $E$, the line $y = x$ and the positive $y$-axis. Using $a = 1$, find the numerical value of the volume of the solid formed when $S$ is rotated through $2\pi$ radians about the $y$-axis

Answer

| | | |
|---|---|---|
| 10 | $x = a\sin\theta \Rightarrow \dfrac{dx}{d\theta} = a\cos\theta$ | |
| | $\int_{\frac{a}{2}}^{a} \sqrt{a^2 - x^2}\ dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{a^2 - (a\sin\theta)^2}\ (a\cos\theta)d\theta$ <br> $= a^2 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\cos^2\theta)d\theta$ | |
| | $= a^2 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \dfrac{(\cos 2\theta + 1)}{2} d\theta$ | |
| | $= \dfrac{a^2}{2}\left[\dfrac{\sin 2\theta}{2} + \theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ | |
| | $= \dfrac{a^2}{2}\left[\dfrac{1}{2}\sin\dfrac{\pi}{2} + \dfrac{\pi}{2} - \dfrac{1}{2}\sin\dfrac{\pi}{3} - \dfrac{\pi}{6}\right] = \dfrac{a^2}{2}\left[\dfrac{\pi}{3} - \dfrac{1}{2}\left(\dfrac{\sqrt{3}}{2}\right)\right]$ | |
| | $= -\dfrac{1}{8}\sqrt{3}a^2 + \dfrac{\pi}{6}a^2 = \dfrac{a^2}{24}(4\pi - 3\sqrt{3})$ (shown) | |
| 10(i) | $x^2 + 3y^2 = a^2 \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{\left(\frac{a}{\sqrt{3}}\right)^2} = 1$ <br>   | |
| 10(ii) | When $x = y$, $x^2 + 3x^2 = a^2 \Rightarrow x = \dfrac{a}{2}$ | |
| | Area of R $= \int_{\frac{a}{2}}^{a} y\ dx + \dfrac{1}{2}\left(\dfrac{a}{2}\right)\left(\dfrac{a}{2}\right) = \dfrac{1}{\sqrt{3}}\int_{\frac{a}{2}}^{a} \sqrt{a^2 - x^2}\ dx + \dfrac{1}{2}\left(\dfrac{a}{2}\right)\left(\dfrac{a}{2}\right)$ | |
| | $= \dfrac{1}{\sqrt{3}}\left[\dfrac{a^2}{24}(4\pi - 3\sqrt{3})\right] + \dfrac{a^2}{8}$ | |
| | $= \dfrac{a^2}{24}\left[\left(\dfrac{4\pi - 3\sqrt{3}}{\sqrt{3}}\right) + 3\right] = \dfrac{a^2}{6}\left(\dfrac{\pi}{\sqrt{3}}\right)$ <br> $k = \dfrac{\sqrt{3}}{18}$ or $k = \dfrac{1}{6\sqrt{3}}$ | |
| 10(iii) | Let $a = 1 \Rightarrow x^2 + 3y^2 = 1$ <br> Vol of S = <br> $\pi\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} x^2\ dy + \dfrac{1}{3}\pi\left(\dfrac{1}{2}\right)^2\left(\dfrac{1}{2}\right)$ <br> $= \pi\int_{\frac{1}{2}}^{\frac{1}{\sqrt{3}}} (1 - 3y^2)\ dy + \dfrac{1}{3}\pi\left(\dfrac{1}{2}\right)^2\left(\dfrac{1}{2}\right)$ | |
| | $= 0.162$ units$^3$ | |

Q6

**Question**

a) Find $\int x\sec^2(x + a)\ dx$, where $a$ is a constant.

b) Find $\int \frac{x-1}{x^2-2x+2}\ dx$

Hence find

(i) The exact value of $\int_1^2 \frac{x-4}{x^2-2x+2} dx$

(ii) $\int_{2-p}^{p} \left|\frac{x-1}{x^2-2x+2}\right| dx$ where $p$ is a constant, $p > 1$. Leave your answer in terms of $p$.

Answer

**8(a)**

$$\int x\sec^2(x+a)\,dx$$
$$= x\tan(x+a) - \int \tan(x+a)\,dx$$
$$= x\tan(x+a) - \ln|\sec(x+a)| + C$$

OR: $x\tan(x+a) + \ln|\cos(x+a)| + C$

**8(b)**

$$\int \frac{x-1}{x^2-2x+2}\,dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x+2}\,dx$$
$$= \frac{1}{2}\ln(x^2-2x+2) + C$$

**8(b) (i)**

$$\int_1^2 \frac{x-4}{x^2-2x+2}\,dx$$
$$= \int_1^2 \frac{x-1}{x^2-2x+2}\,dx - \int_1^2 \frac{3}{x^2-2x+2}\,dx$$
$$= \int_1^2 \frac{x-1}{x^2-2x+2}\,dx - \int_1^2 \frac{3}{(x-1)^2+1}\,dx$$
$$= \frac{1}{2}\left[\ln\left(x^2-2x+2\right)\right]_1^2 - 3\left[\tan^{-1}(x-1)\right]_1^2$$
$$= \frac{1}{2}[\ln 2 - \ln 1] - 3\left[\tan^{-1}1 - \tan^{-1}0\right]$$
$$= \frac{1}{2}\ln 2 - \frac{3\pi}{4}$$

**8(b) (ii)**

Note that $\frac{x-1}{x^2-2x+2} = \frac{x-1}{(x-1)^2+1}$: $-$ | $+$ (1)

$$\int_{2-p}^{p} \left|\frac{x-1}{x^2-2x+2}\right| dx$$
$$= -\int_{2-p}^{1} \frac{x-1}{(x-1)^2+1}\,dx + \int_1^p \frac{x-1}{(x-1)^2+1}\,dx$$
$$= 2\int_1^p \frac{x-1}{(x-1)^2+1}\,dx \quad \text{(by symmetry)}$$
$$= 2\left[\frac{1}{2}\ln(x^2-2x+2)\right]_1^p = \ln\left(p^2-2p+2\right)$$

Q7

**Question**

By using the substitution $u = 1 - x$, show that $\int_0^1 x^n(1-x)^m\,dx = \int_0^1 (1-x)^n x^m\,dx$

Hence, or otherwise, evaluate $\int_0^1 x^2\sqrt{1-x}\,dx$, express your answer in exact form.

Answer

**2.**

From $u = 1 - x$, $\frac{du}{dx} = -1$.

Limits: when $x = 0$, $u = 1$, and when $x = 1$, $u = 0$.

Therefore $\int_0^1 x^n(1-x)^m\,dx = \int_1^0 (1-u)^n u^m\,(-du)$

$$= \int_0^1 (1-u)^n u^m\,du$$
$$= \int_0^1 (1-x)^n x^m\,dx \text{ (by a change of dummy variables)}$$

By substituting $n = 2$ and $m = \frac{1}{2}$ into the previous result:

$$\int_0^1 x^2(1-x)^{\frac{1}{2}}\,dx = \int_0^1 (1-x)^2 x^{\frac{1}{2}}\,dx$$
$$= \int_0^1 \left(1-2x+x^2\right)x^{\frac{1}{2}}\,dx$$
$$= \int_0^1 x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + x^{\frac{5}{2}}\,dx$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}}\right]_0^1 = \frac{16}{105}$$

Q8

**Question**

A curve has equation given by $y = (\ln x)^2 - 1$, where $x > 0$.

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(i) Sketch the graph, indicating the exact coordinates of the $x$-intercepts and the turning point.

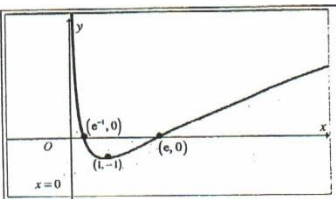
The region $R$ is bounded by the curve and the $x$-axis.

(ii) Find the exact area of $R$

(iii) Find the volume of the solid generated when $R$ is rotated through $2\pi$ radians about the $y$-axis

Answer

11(i)




To obtain $x$-intercepts, let $y=0$

$\Rightarrow (\ln x)^2 = 1$

$\Rightarrow \ln x = \pm 1$

$\Rightarrow x = e^1$ or $e^{-1}$

To obtain the turning point, find $\frac{dy}{dx} = 2\ln x$.

Let $\frac{dy}{dx} = 0 \Rightarrow 2\ln x = 0 \Rightarrow x = 1$

Thus coordinates of turning point is $(1,-1)$.

11(ii) Area of region $R$

$= \int_{e^{-1}}^{e} -\left((\ln x)^2 - 1\right) dx$

$= -\left[x(\ln x)^2\right]_{e^{-1}}^{e} + \int_{e^{-1}}^{e} x\frac{2\ln x}{x} dx + [x]_{e^{-1}}^{e}$

$= -(e-e^{-1}) + 2\left([x\ln x]_{e^{-1}}^{e} - \int_{e^{-1}}^{e} 1\, dx\right) + (e-e^{-1})$

$= -(e-e^{-1}) + 2\left((e+e^{-1}) - (e-e^{-1})\right) + (e-e^{-1})$

$= 4e^{-1}$

11(iii) Make $x$ the subject:

$y = (\ln x)^2 - 1$

$\Rightarrow \ln x = \pm\sqrt{y+1}$

$\Rightarrow x = e^{\pm\sqrt{y+1}}$

Thus the volume obtained

$= \pi\int_{-1}^{0} \left(e^{\sqrt{y+1}}\right)^2 - \left(e^{-\sqrt{y+1}}\right)^2 dy = 12.2$ (to 3 s.f.)

Q9

**Question**

(a) Find $\int \tan^{-1} x \ dx$

(b) Find $\int \frac{2x}{x^2+2x+1} dx$

(c) Use the substitution $x = \frac{1}{u}$ to find the exact value of $\int_1^2 \frac{1}{x\sqrt{x^2-1}} dx$

Answer

| | |
|---|---|
| 7a | $\int \tan^{-1} x \, dx = x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{x^2+1} \, dx + c'$<br>$= x\tan^{-1} x - \frac{1}{2}\ln\left(x^2+1\right) + c$ |
| 7b | $\int \frac{2x}{x^2+2x+1} \, dx = \int \frac{2x+2}{x^2+2x+1} - \frac{2}{(x+1)^2} \, dx$<br>$= \ln\left(x^2+2x+1\right) + \frac{2}{x+1} + c$<br>Alternative solution:<br>$\int \frac{2x}{x^2+2x+1} \, dx = \int \frac{2}{x+1} - \frac{2}{(x+1)^2} \, dx$<br>$= 2\ln(x+1) + \frac{2}{x+1} + c$ |
| 7c | $\int_1^2 \frac{1}{x\sqrt{x^2-1}} \, dx = \int_1^{\frac{1}{2}} \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}-1}}\left(-\frac{1}{u^2}\right) du$<br>$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{1-u^2}} \, du$<br>$= \left[\sin^{-1} u\right]_{\frac{1}{2}}^{1}$<br>$= \frac{\pi}{3}$<br>Alternative solution:<br>$\int_1^2 \frac{1}{x\sqrt{x^2-1}} \, dx = \int_1^{\frac{1}{2}} \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}-1}}\left(-\frac{1}{u^2}\right) du$<br>$= \int_1^{\frac{1}{2}} -\frac{1}{\sqrt{1-u^2}} \, du$<br>$= \left[\cos^{-1} u\right]_1^{\frac{1}{2}}$<br>$= \frac{\pi}{3}$ |

Q10

**Question**

A curve $C$ is defined by the parametric equations $x = \cos t$, $y = \sin 2t$, for $0 \le t \le 2\pi$.

(i) Sketch the curve, stating the coordinates of any points of intersection with the axes.

(ii) Show that the area enclosed by the curve $C$ is $8\int_0^{\frac{\pi}{2}} \sin^2 t \cos t \ dt$. Given that $\int \sin^2 t \ \cos t \ dt = \frac{1}{3}\sin^3 t + c$, find the exact area enclosed by the curve $C$.

(iii) By finding the Cartesian equation of $C$ or otherwise, find the volume of revolution formed when the area enclosed by the curve $C$ is rotated completely about the $x$-axis, giving your answer correct to 3 decimal places.

Answer

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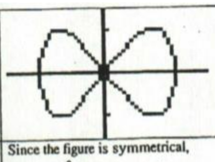
|---|---|
| 8i |  |
| 8ii | Since the figure is symmetrical,<br>$Area = 4\int_{\frac{\pi}{2}}^{0}(\sin 2t)(-\sin t)\ dt$<br>$= 4\int_{0}^{\frac{\pi}{2}} \sin t \sin 2t\ dt$<br>$= 8\int_{0}^{\frac{\pi}{2}} \sin^2 t \cos t\ dt$<br>$Area = 8\left[\frac{\sin^3 t}{3}\right]_0^{\frac{\pi}{2}}$<br>$= \frac{8}{3}$ |
| 8iii | $y = \sin 2t$<br>$= 2\sin t \cos t$<br>$= \pm 2x\sqrt{1-x^2}$<br>$Volume = 2\pi\int_0^1 \left(2x\sqrt{1-x^2}\right)^2\ dx$<br>$= 3.351$<br>*Alternatively,*<br>$Volume = 2\pi\int_{\frac{\pi}{2}}^{0}(\sin 2t)^2(-\sin t)\ dt$<br>$= 3.351$ |

Q11

**Question**

The region enclosed by the curve $y = e^x \sin x$ where $0 \le x \le \frac{\pi}{2}$, the $x$-axis and the line $x = \frac{\pi}{2}$ is denoted by A. Find the exact area of A.

Find the volume of revolution when the region bounded by the curves $y = e^x \sin x$,

$y = x + x^2 + \frac{1}{3}x^3$ and the line $x = \frac{\pi}{2}$ is rotated completely about the $x$-axis.

Answer

| | |
|---|---|
| 2 | Area of $A = \int_0^{\frac{\pi}{2}} e^x \sin x\, dx$<br>$= \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x\, dx$<br>$= e^{\frac{\pi}{2}} - \left\{\left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x\, dx\right\}$<br>$= e^{\frac{\pi}{2}} - \left\{-1 + \int_0^{\frac{\pi}{2}} e^x \sin x\, dx\right\}$<br>$2\int_0^{\frac{\pi}{2}} e^x \sin x\, dx = e^{\frac{\pi}{2}} + 1$<br>$\int_0^{\frac{\pi}{2}} e^x \sin x\, dx = \frac{1}{2}\left(e^{\frac{\pi}{2}} + 1\right)$<br><br>Volume<br>$= \pi\left[\int_0^{\frac{\pi}{2}}\left(x + x^2 + \frac{1}{3}x^3\right)^2 - \int_0^{\frac{\pi}{2}}\left(e^x \sin x\right)^2\right] dx$<br>$\approx 3.19\ units^3$ |

Q12

**Question**

Find $\int \frac{x}{(1+4x^2)^2}\,dx$. Hence find $\int \frac{4x^2}{(1+4x^2)^2}\,dx$

Answer

| | |
|---|---|
| 2 | $\int \frac{x}{(1+4x^2)^2}\,dx = \int \frac{1}{8}\cdot\frac{8x}{(1+4x^2)^2}\,dx$ <br> $= -\frac{1}{8}\left(\frac{1}{1+4x^2}\right)+c$ <br> $\int \frac{4x^2}{(1+4x^2)^2}\,dx = \int 4x\cdot\frac{x}{(1+4x^2)^2}\,dx$ <br> $= 4x\left(-\frac{1}{8(1+4x^2)}\right) - \int 4\left[-\frac{1}{8(1+4x^2)}\right]dx$ <br> $= -\frac{x}{2(1+4x^2)} + \int \frac{1}{2(1+4x^2)}\,dx$ <br> $= -\frac{x}{2(1+4x^2)} + \frac{1}{4}\tan^{-1}2x + c$ |

Q13

**Question**

A curve has parametric equations

$x = \frac{2}{(1-t)^2}$, $y = \frac{t}{1-t}$, where $t \neq 1$.

(i) Points $P$ and $Q$ on $C$ have parameters $p$ and $q$ respectively such that $p < q$. Point $A$ is a point on $C$ when $t = 0$. The normal to $C$ at point $P$ and the normal to $C$ at point $Q$ both pass through point $A$. Find $p$ and $q$.

(ii) Find the area of region bounded by $C$ for $y \geq 0$, the line $x = 8$ and the $x$-axis.

Answer

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11 (i) Point $A(2, 0)$

$\frac{dx}{dt}=\frac{4}{(1-t)^3}$ and $\frac{dy}{dt}=\frac{1}{(1-t)^2}$

$\frac{dy}{dx}=\frac{\frac{1}{(1-t)^2}}{\frac{4}{(1-t)^3}}=\frac{1}{4}(1-t)$

Equation of normal: $y-\frac{t}{1-t}=-\frac{4}{1-t}\left(x-\frac{2}{(1-t)^2}\right)$

Since the normal at $P$ and $Q$ pass through $A$,

$0-\frac{t}{1-t}=-\frac{4}{1-t}\left(2-\frac{2}{(1-t)^2}\right)$

$\frac{8}{(1-t)^3}=\frac{8-t}{1-t}$

Simplifying,

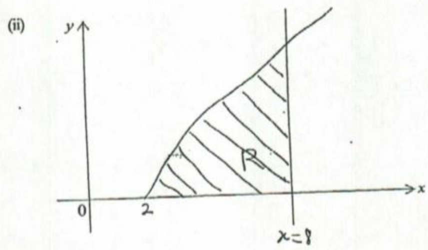
$t^3-10t^2+17t=0$

$t(t^2-10t+17)=0$

$t=0,\quad t^2-10t+17=0 \Rightarrow t=5\pm2\sqrt{2}$

Therefore, $p=5-2\sqrt{2},\ q=5+2\sqrt{2}$

(ii)




When $x=8$,

$\frac{2}{(1-t)^2}=8 \Rightarrow t=\frac{1}{2}$ or $\frac{3}{2}$ (rejected since $y \geq 0$)

Area of region $R$

$=\int_2^8 y\,dx$

$=\int_0^{\frac{1}{2}}\frac{t}{1-t}\left(\frac{4}{(1-t)^3}\right)dt$

$=\int_0^{\frac{1}{2}}\frac{4t}{(1-t)^4}\,dt$

$=\frac{10}{3}$ (using GC)

Q14

**Question**

(i) $\int x^2\, e^x dx$

(ii) Hence, find the exact volume of the solid of revolution formed when the region bounded the curve $y = xe^{\frac{1}{2}x}$, the line $y = 2e, x = 3$ and the $x$-axis is rotated through 4 right angles about the $x$-axis.

Answer

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3 (i)

$$\int x^2e^x\,dx = x^2e^x - 2\int xe^x dx + C' \qquad u = x^2 \quad \frac{dv}{dx} = e^x \quad \frac{du}{dx} = 2x \quad v = e^x$$

$$= x^2e^x - 2\left(xe^x - \int e^x dx\right) + C' \qquad u = x \quad \frac{dv}{dx} = e^x \quad \frac{du}{dx} = 1 \quad v = e^x$$

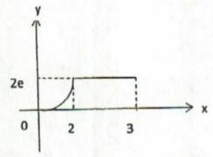
$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= e^x\left(x^2 - 2x + 2\right) + C$$

(ii)

$y = 2e \quad \Rightarrow xe^{\frac{1}{2}x} = 2e$

$\Rightarrow x = 2$ (by observation)




Required volume $= \pi\int_0^2 \left(xe^{\frac{1}{2}x}\right)^2 dx + \pi(2e)^2$

$$= \pi\int_0^2 x^2e^x dx + 4\pi e^2$$

$$= \pi\left[e^x\left(x^2 - 2x + 2\right)\right]_0^2 + 4\pi e^2$$

$$= \pi\left(e^2(4 - 4 + 2) - 2\right) + 4\pi e^2$$

$$= 6\pi e^2 - 2\pi$$

Q15

**Question**

(i) Differentiate $\frac{1}{\sqrt{1-4x^2}}$ with respect to $x$.

(ii) Find $\int \frac{x\sin^{-1}(2x)}{\sqrt{(1-4x^2)^3}}dx$

Answer

| Qn. [Marks] | Solution |
|---|---|
| 1(i) [2] | $\frac{d}{dx}\left(\frac{1}{\sqrt{1-4x^2}}\right) = -\frac{1}{2}\left(1-4x^2\right)^{-\frac{3}{2}}\cdot(-4)(2x)$ $= \frac{4x}{\sqrt{\left(1-4x^2\right)^3}}$ |
| 1(ii) [3] | $\int \frac{x\sin^{-1}(2x)}{\sqrt{\left(1-4x^2\right)^3}}dx$ $= \int \frac{4x}{\sqrt{\left(1-4x^2\right)^3}}\cdot\frac{1}{4}\sin^{-1}(2x)\,dx$ $= \frac{1}{\sqrt{1-4x^2}}\cdot\frac{1}{4}\sin^{-1}(2x) - \int \frac{1}{\sqrt{1-4x^2}}\cdot\frac{1}{4}\cdot\frac{2}{\sqrt{1^2-(2x)^2}}dx$ $= \frac{\sin^{-1}(2x)}{4\sqrt{1-4x^2}} - \frac{1}{4}\int\frac{2}{1^2-(2x)^2}dx$ $= \frac{\sin^{-1}(2x)}{4\sqrt{1-4x^2}} - \frac{1}{8}\ln\left\lvert\frac{1+2x}{1-2x}\right\rvert + C$ or $\frac{\sin^{-1}(2x)}{4\sqrt{1-4x^2}} - \frac{1}{8}\ln\frac{1+2x}{1-2x} + C$ |

Q16

**Question**

(i) Find $\int x\sin x\;dx$.

(ii) Sketch the graph $y = x\sin x$ for $0 \le x \le k\pi$ where $1 < k < \frac{3}{2}$

It is given that $\int_0^{k\pi} |\, x \sin x \,|\ dx = \frac{4}{3}\pi + \frac{\sqrt{3}}{2}$ where $k$ is a constant such that $1 < k < \frac{3}{2}$. Find the exact value of $k$.

Answer

**Question 3 [8 Marks]**

i

$u = x \qquad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1 \qquad v = -\cos x$

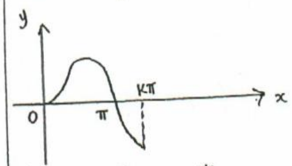
$\int x \sin x \, dx = -x\cos x + \int \cos x \, dx$

$= -x\cos x + \sin x + c$

ii

$\int_0^{k\pi} |x \sin x| \, dx = \frac{4}{3}\pi + \frac{\sqrt{3}}{2}$

Using GC to observe the graph of $y = x\sin x$, we see that it is above the $x$-axis from 0 to $\pi$ and below from $\pi$ to $k\pi$ where $1 < k < \frac{3}{2}$.




$\int_0^{k\pi} |x \sin x| \, dx = \int_0^{\pi} x \sin x \, dx + \int_{\pi}^{k\pi} -x \sin x \, dx$

$= [-x\cos x + \sin x]_0^{\pi} - [-x\cos x + \sin x]_{\pi}^{k\pi}$

$= \pi - [-k\pi\cos(k\pi) + \sin(k\pi) - \pi]$

$= 2\pi + k\pi\cos(k\pi) - \sin(k\pi)$

$= [2 + k\cos(k\pi)]\pi - \sin(k\pi) = \frac{4}{3}\pi + \frac{\sqrt{3}}{2}$

Comparing the term without the factor $\pi$,

$\sin(k\pi) = -\frac{\sqrt{3}}{2}$

$\Rightarrow k\pi = \frac{4\pi}{3}, \frac{5\pi}{3}$

$\Rightarrow k = \frac{4}{3}$ or $\frac{5}{3}$

Since $1 < k < \frac{3}{2}$, $\therefore k = \frac{4}{3}$

Q17

**Question**

It is given that $f(x) = \begin{cases} ax & for\ 0 \le x \le a \\ 2a^2 - ax & for\ a < x < 2a \end{cases}$

And that $f(x + 2a) = \frac{1}{2} f(x)$ for all real values of $x$ where $a$ is a positive real constant.

(i) Sketch the graph of $y = f(x)$ for $-2a \le x \le 4a$

(ii) Show that the exact value of $\int_0^{2a} f(x)\, dx = ka^3$, where $k$ is a constant to be determined.

(iii) Hence, evaluate exactly, in forms of $a$, $\int_0^{\infty} f(x)\ dx$.

Answer

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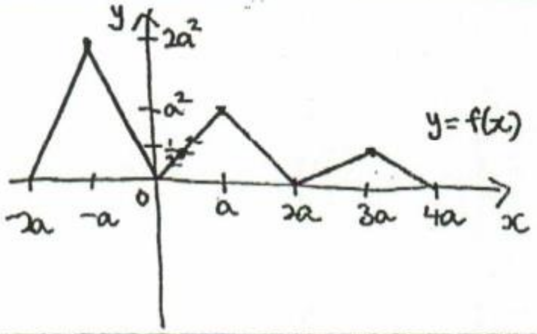
**Question 7 [7 Marks]**

i




ii

$$\int_0^{2a} f(x)\,dx = \int_0^{a} ax\,dx + \int_a^{2a} 2a^2 - ax\,dx$$

$$= \left[\frac{ax^2}{2}\right]_0^a + \left[2a^2x - \frac{ax^2}{2}\right]_a^{2a}$$

$$= \frac{a^3}{2} + \left[4a^3 - \frac{4a^3}{2} - 2a^3 + \frac{a^3}{2}\right]$$

$$= a^3$$

So $k = 1$.

**Alternatively,**

$$\int_0^{2a} f(x)\,dx = \frac{1}{2}(2a)(a^2)$$

$$= a^3$$

So $k = 1$.

iii

$$\int_0^{\infty} f(x)\,dx$$

$$= \frac{1}{2}(2a)(a^2) + \frac{1}{2}(2a)\left(\frac{1}{2}a^2\right) + \frac{1}{2}(2a)\left(\left(\frac{1}{2}\right)^2 a^2\right) + \dots$$

$$= a^3\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots\right)$$

$$= a^3 \cdot \frac{1}{1 - \frac{1}{2}}$$

$$= 2a^3$$

Q18

**Question**

A curve C has parametric equation

$$x = 1 - \cos t, \quad y = \frac{1}{2}\sin(2t), \qquad \text{for } \ 0 \le t \le \frac{\pi}{2}$$

(i) Sketch $C$, stating the coordinates of any points of intersection with the axes.

(ii) Find the equation of the normal to $C$ at the point where $t = \frac{\pi}{3}$.

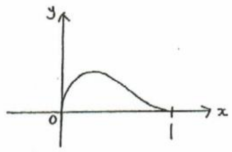
(iii) The region $R$ is bounded by the part of the curve $C$ where $0 \le t \le \frac{\pi}{6}$, the $x$-axis, and the vertical line $x = \alpha$ where $\alpha = 1 - \cos\frac{\pi}{6}$. Find the exact area of $R$.

(iv) Determine a Cartesian equation of $C$, and use it to find the numerical value of the volume of revolution when $R$ is rotated completely about the $x$-axis.

Answer

**Question 12 [12 Marks]**

i




ii

$$x = 1 - \cos t \Rightarrow \frac{dx}{dt} = \sin t$$

$$y = \frac{1}{2}\sin(2t) \Rightarrow \frac{dy}{dt} = \frac{1}{2}(2)\cos(2t) = \cos(2t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos(2t)}{\sin t}$$

At $t = \frac{\pi}{3}$,

$$x = 1 - \cos\frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2} \text{ and}$$

$$y = \frac{1}{2}\sin\left(2\cdot\frac{\pi}{3}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$$

$$\frac{dy}{dx} = \frac{\cos\left(\frac{2\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

So, equation of normal is

$$y - \frac{\sqrt{3}}{4} = \sqrt{3}\left(x - \frac{1}{2}\right)$$

$$y - \frac{\sqrt{3}}{4} = \sqrt{3}x - \frac{\sqrt{3}}{2}$$

$$\therefore y = \sqrt{3}x - \frac{\sqrt{3}}{4}$$

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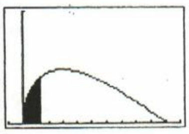
iii



<u>Method 1:</u>

Area of $R$

$$=\int_0^{\pi} y\,dx$$

$$=\int_0^{\frac{\pi}{6}} \frac{1}{2}\sin(2t)\cdot \sin t\, dt$$

$$=\int_0^{\frac{\pi}{6}} \sin t \cos t \cdot \sin t\, dt$$

$$=\int_0^{\frac{\pi}{6}} \cos t\,(\sin t)^2\, dt$$

$$=\left[\frac{1}{3}(\sin t)^3\right]_0^{\frac{\pi}{6}}$$

$$=\frac{1}{3}\left(\sin\frac{\pi}{6}\right)^3-\frac{1}{3}(\sin 0)^3$$

$$=\frac{1}{3}\left(\frac{1}{2}\right)^3-0$$

$$=\frac{1}{24}\text{ units}^2$$

<u>Method 2:</u>

Area of $R$

$$=\int_0^{\alpha} y\,dx$$

$$=\int_0^{\frac{\pi}{6}} \frac{1}{2}\sin(2t)\cdot \sin t\, dt$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{6}} \sin(2t)\cdot \sin t\,dt$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{6}}\left(-\frac{1}{2}\right)(\cos 3t-\cos t)\,dt$$

$$=-\frac{1}{4}\left[\frac{1}{3}\sin 3t-\sin t\right]_0^{\frac{\pi}{6}}$$

$$=-\frac{1}{4}\left(\frac{1}{3}\sin\frac{3\pi}{6}-\sin\frac{\pi}{6}\right)$$

$$=-\frac{1}{4}\left(\frac{1}{3}-\frac{1}{2}\right)$$

$$=\frac{1}{24}\text{ units}^2$$

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iv

Finding Cartesian equation:

<u>Method 1</u>

$x = 1-\cos t \Rightarrow \cos t = 1-x \Rightarrow t = \cos^{-1}(1-x)$

$y = \frac{1}{2}\sin(2t)$

$y = \frac{1}{2}\sin\left(2\cos^{-1}(1-x)\right)$

---

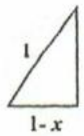
<u>Method 2</u>

$x = 1-\cos t$

$\Rightarrow \cos t = 1-x$

$\Rightarrow \sin t = \sqrt{1^2-(1-x)^2}$

$\Rightarrow \sin t = \sqrt{2x-x^2}$




$y = \frac{1}{2}\sin(2t)$

$y = \sin t \cos t$

$\therefore y = \sqrt{2x-x^2}\cdot(1-x)$

---

<u>Method 3</u>

$x = 1-\cos t \Rightarrow \cos t = 1-x$

$y = \frac{1}{2}\sin(2t) = \sin t \cos t$

$\Rightarrow y = \sin t\cdot(1-x)$

$\Rightarrow \frac{y}{1-x} = \sin t$

$\sin^2 t + \cos^2 t = 1$

$\Rightarrow \left(\frac{y}{1-x}\right)^2 + (1-x)^2 = 1$

$\therefore y^2 = (1-x)^2 - (1-x)^4$

---

Required volume

$= \pi\int_0^{\alpha} y^2\, dx$

$= \pi\int_0^{1-\cos\frac{\pi}{6}} y^2\, dx$

$= \pi\int_0^{1-\cos\frac{\pi}{6}} y^2\, dx$

$= 0.0447829016$

$= 0.0448$ (3 s.f)