Q1

Question

By using the substitution $u = 1 - x$, show that $\int_0^1 x^n(1-x)^m\,dx = \int_0^1 (1-x)^n x^m\,dx$

Hence, or otherwise, evaluate $\int_0^1 x^2\sqrt{1-x}\,dx$, express your answer in exact form.

Answer

2\. From $u = 1 - x$, $\frac{du}{dx} = -1$.

Limits: when $x = 0$, $u = 1$, and when $x = 1$, $u = 0$.

Therefore $\int_0^1 x^n(1-x)^m\,dx = \int_1^0 (1-u)^n u^m\,(-du)$

$= \int_0^1 (1-u)^n u^m\,du$

$= \int_0^1 (1-x)^n x^m\,dx$ (by a change of dummy variables)

By substituting $n = 2$ and $m = \frac{1}{2}$ into the previous result:

$\int_0^1 x^2(1-x)^{\frac{1}{2}}\,dx = \int_0^1 (1-x)^2 x^{\frac{1}{2}}\,dx$

$= \int_0^1 (1-2x+x^2)x^{\frac{1}{2}}\,dx$

$= \int_0^1 x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + x^{\frac{5}{2}}\,dx$

$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{4}{5}x^{\frac{5}{2}} + \frac{2}{7}x^{\frac{7}{2}}\right]_0^1 = \frac{16}{105}$.

Q2

Question

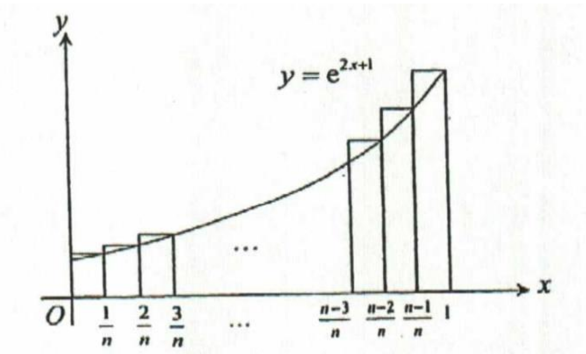
(a)

(i) Show that $(k+1)!\,k - k!\,(k-1) = k!\,(k^2+1)$.

(ii) Hence find $\sum_{k=1}^{n} k!\,(k^2+1)$

(iii) Using your answer in part (ii), find $\sum_{k=1}^{n-1}(k+1)!\,(k^2+2k+2)$.

(b) The graph of $y = e^{2x+1}$, for $0 \le x \le 1$, is shown in the diagram. Rectangles of equal width are drawn as shown in the interval between $x = 0$ and $x = 1$.



(i) Show that the total area of all the n rectangles, A, is given by $\frac{e}{n}\left(\frac{e^{\frac{2}{n}}(1-e^2)}{1-e^{\frac{2}{n}}}\right)$.

(ii) By considering the area under the curve $y = e^{2x+1}$, find the exact value of the limit of A as $n \to \infty$

(iii) Hence show that $\frac{e^{\frac{2}{n}}}{n\left(e^{\frac{2}{n}}-1\right)} > k$, where k is a constant to be found. Find the largest possible value of k.

Answer

11 Summation & Integration

ai

$$\text{LHS} = (k+1)!k - k!(k-1)$$
$$= k!\left(k(k+1)-(k-1)\right)$$
$$= k!(k^2+1) = \text{RHS}$$

(ii)

$$\sum_{k=1}^{n} k!(k^2+1) = \sum_{k=1}^{n}\left[(k+1)!k - k!(k-1)\right]$$
$$= 2!(1) - 1!(0)$$
$$+3!(2) - 2!(1)$$
$$+4!(3) - 3!(2)$$
$$+\ldots$$
$$+(n-1)!(n-2) - (n-2)!(n-3)$$
$$+n!(n-1) - (n-1)!(n-2)$$
$$+(n+1)!n - n!(n-1)$$
$$= (n+1)!n$$

(iii)

$$\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2) = \sum_{k-1=1}^{k-1=n-1}(k-1+1)!\left((k-1)^2+2(k-1)+2\right) \quad \text{(replace } k \text{ by } k-1\text{)}$$
$$= \sum_{k=2}^{n} k!(k^2+1)$$
$$= \sum_{k=1}^{n} k!(k^2+1) - 1!(1^2+1)$$
$$= (n+1)!n - 2$$

Alternative Method: (Strongly not recommended)

Result: $\sum_{k=1}^{n} k!(k^2+1) = (n+1)!n$

Replace k by $k+1$,

$$\sum_{k+1=1}^{k+1=n}(k+1)!\left((k+1)^2+1\right) = (n+1)!n$$
$$\Rightarrow \sum_{k=0}^{n-1}(k+1)!(k^2+2k+2) = (n+1)!n$$
$$\therefore \sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)$$
$$= \sum_{k=0}^{n-1}(k+1)!(k^2+2k+2) - (0+1)!(0^2+2(0)+2)$$
$$= \sum_{k=1}^{n} k!(k^2+1) - 2$$
$$= (n+1)!n - 2$$

(b) (i)

$$A = \frac{1}{n}\left[e^{\frac{2}{n}+1} + e^{\frac{4}{n}+1} + \ldots + e^{\frac{2(n-1)}{n}+1} + e^3\right]$$
$$= \frac{e}{n}\left[e^{\frac{2}{n}} + e^{\frac{4}{n}} + \ldots + e^{\frac{2(n-1)}{n}} + e^2\right]$$
$$= \frac{e}{n}\left[\frac{e^{\frac{2}{n}}\left(1-\left(e^{\frac{2}{n}}\right)^n\right)}{1-e^{\frac{2}{n}}}\right] \quad \text{(Applying GP sum formula)}$$
$$= \frac{e}{n}\left[\frac{e^{\frac{2}{n}}\left(1-e^2\right)}{1-e^{\frac{2}{n}}}\right] \quad \text{(shown)}$$

(ii)

As $n \to \infty$,

$$A \to \int_0^1 e^{2x+1}\,dx = \left[\frac{1}{2}e^{2x+1}\right]_0^1$$
$$= \frac{1}{2}e^3 - \frac{1}{2}e$$
$$= \frac{e}{2}\left(e^2-1\right)$$

(iii)

Since the sum of the area of rectangles is an overestimate,

$$\frac{e}{n}\left[\frac{e^{\frac{2}{n}}\left(1-e^2\right)}{1-e^{\frac{2}{n}}}\right] > \frac{e}{2}\left(e^2-1\right)$$
$$\Rightarrow \frac{e}{n}\left[\frac{e^{\frac{2}{n}}\left(e^2-1\right)}{e^{\frac{2}{n}}-1}\right] > \frac{e}{2}\left(e^2-1\right)$$
$$\Rightarrow \frac{e^{\frac{2}{n}}}{n\left(e^{\frac{2}{n}}-1\right)} > \frac{1}{2} \quad \left(\div e\left(e^2-1\right) \text{ on both sides since } e\left(e^2-1\right)>0\right)$$
$$\therefore k = \frac{1}{2}$$

Q3

Question

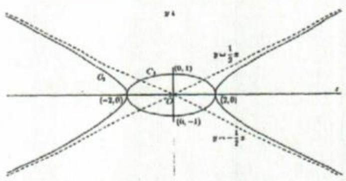
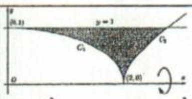
The curve C_1 has equation $\frac{x^2}{4} + y^2 = 1$. The curve C_2 has equation $\frac{x^2}{4} - y^2 = 1$.

(i) Sketch C_1 and C_2 on the same diagram, labelling clearly the exact coordinates of the point(s) of intersection with the axes and the equation(s) of the asymptote(s), if any.

(ii) Find the volume of revolution when the region bounded by C_1, C_2 and the line $y = 1$ for $x \geq 0$ is rotated completely about the x-axis. Give your answer correct to 4 decimal places.

(iii) By using a substitution of the form $x = a\cos\theta$, where a is a positive constant and $0 \leq \theta \leq \frac{\pi}{2}$, find the exact area bounded by C_1, the positive x-axis and the line $x = 1$.

Answer

12 (i)	
12 (ii)	 C_1: $\frac{x^2}{4} + y^2 = 1 \Rightarrow y^2 = 1 - \frac{x^2}{4}$ C_2: $\frac{x^2}{4} - y^2 = 1 \Rightarrow y^2 = \frac{x^2}{4} - 1$ Volume generated $= \pi(1^2)(\sqrt{8}) - \pi\int_0^2 \left(1 - \frac{x^2}{4}\right) dx - \pi\int_2^{\sqrt{8}} \left(\frac{x^2}{4} - 1\right) dx$ $= 3.4701$ (to 4 dec places)
12 (iii)	$\frac{x^2}{4} + y^2 = 1 \Rightarrow y = \sqrt{1 - \frac{x^2}{4}}$ Using the substitution $x = 2\cos\theta$, $\frac{dx}{d\theta} = -2\sin\theta$. When $x = 1, \theta = \frac{\pi}{3}$; when $x = 2, \theta = 0$. Area of region $= \int_1^2 \sqrt{1 - \frac{x^2}{4}}\, dx = \int_{\frac{\pi}{3}}^0 \sqrt{1 - \frac{4\cos^2\theta}{4}}(-2\sin\theta)\, d\theta$ $= \int_0^{\frac{\pi}{3}} 2\sin^2\theta\, d\theta = \int_0^{\frac{\pi}{3}} 1 - \cos 2\theta\, d\theta$ $= \left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{3}}$ $= \left[\frac{\pi}{3} - \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right] - \left[0 - \frac{1}{2}\sin(0)\right]$ $= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

Q4

Question

It is given that $f(x) = \begin{cases} ax & for\ 0 \leq x \leq a \\ 2a^2 - ax & for\ a < x < 2a \end{cases}$

And that $f(x + 2a) = \frac{1}{2}f(x)$ for all real values of x where a is a positive real constant.

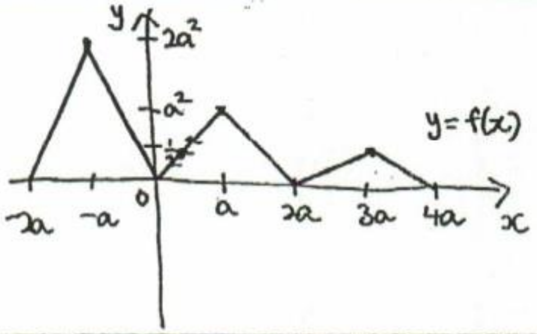
(i) Sketch the graph of $y = f(x)$ for $-2a \leq x \leq 4a$

(ii) Show that the exact value of $\int_0^{2a} f(x)\, dx = ka^3$, where k is a constant to be determined.

(iii) Hence, evaluate exactly, in forms of a, $\int_0^{\infty} f(x)\ dx$.

Answer

Question 7 [7 Marks]

i	
ii	$\int_0^{2a} f(x)\,dx = \int_0^a ax\,dx + \int_a^{2a} 2a^2 - ax\,dx$ $= \left[\frac{ax^2}{2}\right]_0^a + \left[2a^2x - \frac{ax^2}{2}\right]_a^{2a}$ $= \frac{a^3}{2} + \left[4a^3 - \frac{4a^3}{2} - 2a^3 + \frac{a^3}{2}\right]$ $= a^3$ So $k = 1$. **Alternatively,** $\int_0^{2a} f(x)\,dx = \frac{1}{2}(2a)(a^2)$ $= a^3$ So $k = 1$.
iii	$\int_0^{\infty} f(x)\,dx$ $= \frac{1}{2}(2a)(a^2) + \frac{1}{2}(2a)\left(\frac{1}{2}a^2\right) + \frac{1}{2}(2a)\left(\left(\frac{1}{2}\right)^2 a^2\right) + \ldots$ $= a^3\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots\right)$ $= a^3 \cdot \frac{1}{1 - \frac{1}{2}}$ $= 2a^3$

Q5

Question

A curve C has parametric equation

$$x = 1 - \cos t, \quad y = \frac{1}{2}\sin(2t), \qquad \text{for} \quad 0 \le t \le \frac{\pi}{2}$$

(i) Sketch C, stating the coordinates of any points of intersection with the axes.

(ii) Find the equation of the normal to C at the point where $t = \frac{\pi}{3}$.

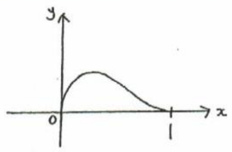
(iii) The region R is bounded by the part of the curve C where $0 \le t \le \frac{\pi}{6}$, the x-axis, and the vertical line $x = \alpha$ where $\alpha = 1 - \cos\frac{\pi}{6}$. Find the exact area of R.

(iv) Determine a Cartesian equation of C, and use it to find the numerical value of the volume of revolution when R is rotated completely about the x-axis.

Answer

Question 12 [12 Marks]

i



ii

$$x = 1 - \cos t \Rightarrow \frac{dx}{dt} = \sin t$$

$$y = \frac{1}{2}\sin(2t) \Rightarrow \frac{dy}{dt} = \frac{1}{2}(2)\cos(2t) = \cos(2t)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos(2t)}{\sin t}$$

At $t = \frac{\pi}{3}$,

$$x = 1 - \cos\frac{\pi}{3} = 1 - \frac{1}{2} = \frac{1}{2} \text{ and}$$

$$y = \frac{1}{2}\sin\left(2\cdot\frac{\pi}{3}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$$

$$\frac{dy}{dx} = \frac{\cos\left(\frac{2\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}}$$

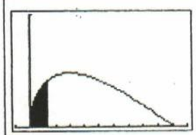
So, equation of normal is

$$y - \frac{\sqrt{3}}{4} = \sqrt{3}\left(x - \frac{1}{2}\right)$$

$$y - \frac{\sqrt{3}}{4} = \sqrt{3}x - \frac{\sqrt{3}}{2}$$

$$\therefore y = \sqrt{3}x - \frac{\sqrt{3}}{4}$$

iii



Method 1:

Area of R

$$= \int_0^{\alpha} y\,dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{2}\sin(2t)\cdot\sin t\,dt$$

$$= \int_0^{\frac{\pi}{6}} \sin t\cos t\cdot\sin t\,dt$$

$$= \int_0^{\frac{\pi}{6}} \cos t(\sin t)^2\,dt$$

$$= \left[\frac{1}{3}(\sin t)^3\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{3}\left(\sin\frac{\pi}{6}\right)^3 - \frac{1}{3}(\sin 0)^3$$

$$= \frac{1}{3}\left(\frac{1}{2}\right)^3 - 0$$

$$= \frac{1}{24}\text{ units}^2$$

Method 2:

Area of R

$$= \int_0^{\alpha} y\,dx$$

$$\doteq \int_0^{\frac{\pi}{6}} \frac{1}{2}\sin(2t)\cdot\sin t\,dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin(2t)\cdot\sin t\,dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} \left(-\frac{1}{2}\right)(\cos 3t - \cos t)\,dt$$

$$= -\frac{1}{4}\left[\frac{1}{3}\sin 3t - \sin t\right]_0^{\frac{\pi}{6}}$$

$$= -\frac{1}{4}\left(\frac{1}{3}\sin\frac{3\pi}{6} - \sin\frac{\pi}{6}\right)$$

$$= -\frac{1}{4}\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{1}{24}\text{ units}^2$$

iv

Finding Cartesian equation:

<u>Method 1</u>

$x=1-\cos t \Rightarrow \cos t=1-x \Rightarrow t=\cos^{-1}(1-x)$

$y=\frac{1}{2}\sin(2t)$

$y=\frac{1}{2}\sin\left(2\cos^{-1}(1-x)\right)$

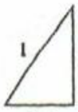
<u>Method 2</u>

$x=1-\cos t$

$\Rightarrow \cos t=1-x$

$\Rightarrow \sin t=\sqrt{1^2-(1-x)^2}$

$\Rightarrow \sin t=\sqrt{2x-x^2}$



1

1- x

$y=\frac{1}{2}\sin(2t)$

$y=\sin t\cos t$

$\therefore y=\sqrt{2x-x^2}\cdot(1-x)$

<u>Method 3</u>

$x=1-\cos t \Rightarrow \cos t=1-x$

$y=\frac{1}{2}\sin(2t)=\sin t\cos t$

$\Rightarrow y=\sin t\cdot(1-x)$

$\Rightarrow \frac{y}{1-x}=\sin t$

$\sin^2 t+\cos^2 t=1$

$\Rightarrow \left(\frac{y}{1-x}\right)^2+(1-x)^2=1$

$\therefore y^2=(1-x)^2-(1-x)^4$

Required volume

$=\pi\int_0^{\alpha} y^2\,dx$

$=\pi\int_0^{1-\cos\frac{\pi}{6}} y^2\,dx$

$=\pi\int_0^{1-\cos\frac{\pi}{6}} y^2\,dx$

$=0.0447829016$

$=0.0448$ (3 s.f)

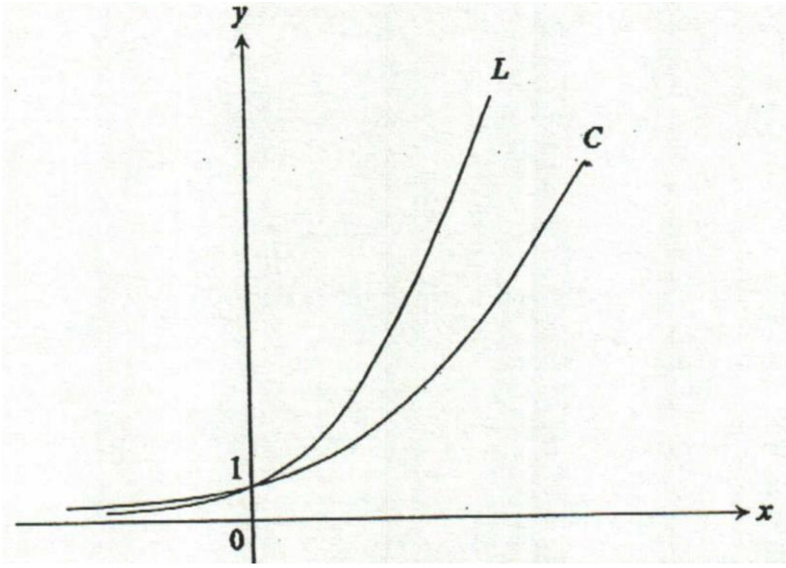
Q6

Question

(a) The curve C is defined by the parametric equations

$x=\ln t$, $y=\frac{t^3+t}{t+1}$, where $t>0$.

Another curve L is defined by the equation $y=e^{2x}$. The graphs of C and L are shown in the diagram below.



Find the exact area of the region bounded by C, L and the line $x = \ln 2$, giving your answer in the form $\ln b$ where b is a constant to be determined.

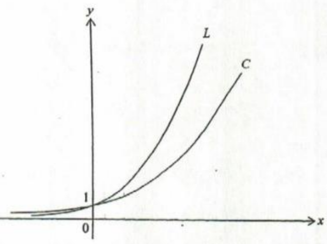
(b) The curves V and W have equations $2y = (x - 1)^2 + 4$ and $y = 2x^2$ Respectively. The region in the first quadrant enclosed by the curves and the y-axis is denoted by S.
Find the exact volume of the solid generated when the region S is rotated through 2π radians about the y-axis.

Answer

10 (a) The curve C is defined by the parametric equations

$$x = \ln t, \quad y = \frac{t^3 + t}{t+1} \qquad \text{where } t > 0.$$

Another curve L is defined by the equation $y = e^{2x}$. The graphs of C and L are shown in the diagram below.



Find the exact area of the region bounded by C, L and the line $x = \ln 2$, giving your answer in the form $\ln b$ where b is a constant to be determined. [5]

(b) The curves V and W have equations $2y = (x-1)^2 + 4$ and $y = 2x^2$ respectively. The region in the first quadrant enclosed by the curves and the y-axis is denoted by S.
Find the exact volume of the solid generated when the region S is rotated through 2π radians about the y-axis. [4]

Solution

(a) $\frac{dx}{dt} = \frac{1}{t}$

Area of the region $= \int_0^{\ln 2} e^{2x}\,dx - \int_0^{\ln 2} y\,dx$

$= \int_0^{\ln 2} e^{2x}\,dx - \int_1^2 \left(\frac{t^3+t}{t+1}\right)\left(\frac{1}{t}\right)dt$

$= \left[\frac{1}{2}e^{2x}\right]_0^{\ln 2} - \int_1^2 \frac{t^2+1}{t+1}\,dt$

$= \left[\frac{1}{2}e^{2x}\right]_0^{\ln 2} - \int_1^2 t - 1 + \frac{2}{t+1}\,dt$

$= \left[\frac{1}{2}e^{2x}\right]_0^{\ln 2} - \left[\frac{t^2}{2} - t + 2\ln(t+1)\right]_1^2$

$= \left(\frac{1}{2}e^{2\ln 2} - \frac{1}{2}\right) - \left[\left(\frac{2^2}{2} - 2 + 2\ln(2+1)\right) - \left(\frac{1^2}{2} - 1 + 2\ln(1+1)\right)\right]$

$= \left(2 - \frac{1}{2}\right) - \left[(2 - 2 + 2\ln 3) - \left(\frac{1}{2} - 1 + 2\ln 2\right)\right]$

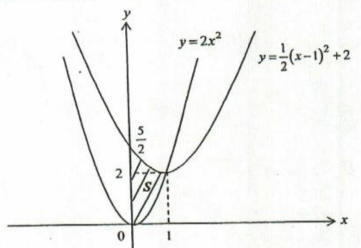
$= \frac{3}{2} - 2\ln 3 - \frac{1}{2} + 2\ln 2$

$= 1 + 2\ln\frac{2}{3}$

$= \ln e + \ln\frac{4}{9}$

$= \ln\left(\frac{4e}{9}\right)$ where $b = \frac{4e}{9}$

(b)



Required Volume $= \pi\left[\int_0^2 \frac{y}{2}\,dy + \int_2^{\frac{5}{2}}\left(1-\sqrt{2y-4}\right)^2 dy\right]$

$= \pi\left\{\left[\frac{y^2}{4}\right]_0^2 + \int_2^{\frac{5}{2}}\left(1-2\sqrt{2y-4}+2y-4\right)dy\right\}$

$= \pi + \pi\int_2^{\frac{5}{2}}\left(-2\sqrt{2y-4}+2y-3\right)dy$

$= \pi + \pi\left[\frac{-2(2y-4)^{\frac{3}{2}}}{2\left(\frac{3}{2}\right)} + y^2 - 3y\right]_2^{\frac{5}{2}}$

$= \pi + \pi\left[-\frac{2}{3}(2y-4)^{\frac{3}{2}} + y^2 - 3y\right]_2^{\frac{5}{2}}$

$= \pi + \pi\left[\left(-\frac{2}{3}\left(2\left(\frac{5}{2}\right)-4\right)^{\frac{3}{2}} + \left(\frac{5}{2}\right)^2 - 3\left(\frac{5}{2}\right)\right) - \left(-\frac{2}{3}(2(2)-4)^{\frac{3}{2}} + (2)^2 - 3(2)\right)\right]$

$= \pi + \frac{\pi}{12}$

$= \frac{13\pi}{12}$ cubic units

Q7

Question

(a)

(i) By using a graphic calculator, find the x-coordinates of the points of intersection of the curves $y = e^x$ and $y = 2x + 1$. Hence solve the inequality $e^x < 2x + 1$
Hence solve the inequality $e^x < 2x + 1$.

(ii) Hence, find the exact value of $\int_{-2}^{1} |e^x - 2x - 1|\ dx$

(b) Find $\int \frac{2x+1}{x^2-4x+7} dx$

(c) Find $\int (\sin x)\ln(\cos x)\ dx$

Answer

2 (a) (i) By using a graphic calculator, find the x-coordinates of the points of intersection of the curves $y = e^x$ and $y = 2x+1$.
Hence solve the inequality $e^x < 2x+1$. [2]

(ii) Hence, find the exact value of $\int_{-2}^{1}|e^x-2x-1|\,dx$. [3]

(b) Find $\int \frac{2x+1}{x^2-4x+7}dx$. [3]

(c) Find $\int \sin x \ln(\cos x)\,dx$. [2]

Solution

(ai) $x = 0$ or $x = 1.26$

$0 < x < 1.26$

$e^x - 2x - 1 < 0$ for $0 < x < 1$ and $e^x - 2x - 1 > 0$ for $-2 < x < 0$

(ii) $\int_{-2}^{1}|e^x-2x-1|dx = \int_{-2}^{0}(e^x-2x-1)dx - \int_{0}^{1}(e^x-2x-1)dx$

$= \left[e^x - x^2 - x\right]_{-2}^{0} - \left[e^x - x^2 - x\right]_0^1$

$= 6 - e^{-2} - e$

(b) $\int \frac{2x+1}{x^2-4x+7}dx = \int \frac{2x-4+5}{x^2-4x+7}dx$

$= \int \frac{2x-4}{x^2-4x+7}dx + \int \frac{5}{(x-2)^2+3}dx$

$= \ln\left(x^2-4x+7\right) + \frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + c$

(c) $\int \sin x \ln(\cos x)\ dx = -\cos x \ln(\cos x) - \int(-\cos x)\left(\frac{-\sin x}{\cos x}\right) dx$

$= -\cos x \ln(\cos x) - \int \sin x\, dx$

$= -\cos x \ln(\cos x) + \cos x + C$