Q1

Question

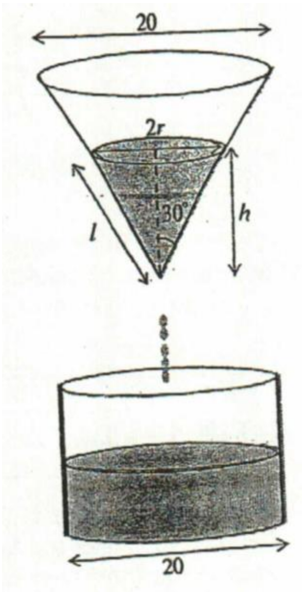
An experiment is conducted using the conical filter which is held with its axis vertical as shown. The filter has a radius of 10cm and semi-vertical angle $30°$. Chemical solution flows from the filter into the cylindrical container, with radius 10cm, at a constant rate of 3 cm^3/s. At time t seconds, the amount of solution in the filter has height h cm and radius r cm.

(i) Find the rate of decrease of radius of the solution in the filter when $h = 5cm.$

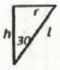
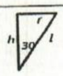
(ii) Let S denotes the curved surface area of filter in contact with the solution. Show that $\frac{dS}{dt} = -\frac{4\sqrt{3}}{r}\ cm^2/s.$

(iii) When the height of solution in the cylindrical container measures 0.81 cm, volumes of solution left in the filter and the container are the same. Find the rate of change of S at this instant.

[The volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$ and the curved surface area is $\pi r l$ where l is the slant height of the cone.]



Solution

4(i)	$V=\frac{1}{3}\pi r^2 h \Rightarrow V=\frac{1}{3}\pi r^2\left(\sqrt{3}r\right)=\frac{1}{\sqrt{3}}\pi r^3$  $\frac{r}{h}=\tan 30^\circ=\frac{1}{\sqrt{3}}$ $\Rightarrow \frac{dV}{dt}=\sqrt{3}\pi r^2\left(\frac{dr}{dt}\right)$ $\Rightarrow \frac{dr}{dt}=\frac{1}{\sqrt{3}\pi r^2}\left(\frac{dV}{dt}\right)=\frac{-3}{\sqrt{3}\pi r^2}$ When $h=5, r=\frac{5}{\sqrt{3}}$ $\Rightarrow \frac{dr}{dt}=\frac{-3}{\sqrt{3}\pi r^2}=\frac{-3\sqrt{3}}{25\pi}$ cm/s $=0.0662$ cm/s.
4(ii)	$S=\pi rl \Rightarrow S=\pi r\left(\frac{r}{\sin 30^\circ}\right)=2\pi r^2$  $\frac{r}{l}=\sin 30^\circ=\frac{1}{2}$ $S=2\pi r^2 \Rightarrow \frac{dS}{dr}=4\pi r$ By chain rule, $\frac{dS}{dt}=\frac{dS}{dr}\times\frac{dr}{dt}$ $\frac{dS}{dt}=4\pi r\times\frac{-3}{\sqrt{3}\pi r^2}=\frac{-4\sqrt{3}}{r}$
4(iii)	Cylinder : $V=\pi r^2 h=\pi(10)^2(0.81)=81\pi$ cone : $V=\frac{1}{\sqrt{3}}\pi r^3$ $\Rightarrow 81\pi=\frac{1}{\sqrt{3}}\pi r^3$ $\Rightarrow r=\left(81\sqrt{3}\right)^{1/3}=3\sqrt{3}$ $\frac{dS}{dt}=\frac{-4\sqrt{3}}{3\sqrt{3}}=-\frac{4}{3}$ cm²/s

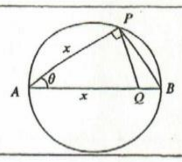
Q2

Question

P is a variable point on the circumference of a circle with diameter AB, and Q is the point on AB such that $AQ = AP$. Given that angle $PAQ = \theta$ radians and $AB = l$, show that the area S of triangle PAQ is given by $S = \frac{1}{2}l^2(\sin\theta - \sin^3\theta)$.

Use differentiation to find, in surd form and in terms of l, the maximum value of S, proving that it is a maximum.

Solution

2	

$$\text{In } \Delta PAB,\ \cos\theta = \frac{AP}{AB} = \frac{x}{\ell} \quad \therefore\ x = \ell\cos\theta$$

$$\begin{aligned} S &= \frac{1}{2}x^2\sin\theta \\ &= \frac{1}{2}(\ell\cos\theta)^2\sin\theta \\ &= \frac{1}{2}\ell^2\left(1-\sin^2\theta\right)\sin\theta \\ &= \frac{1}{2}\ell^2\left(\sin\theta-\sin^3\theta\right) \quad \text{(shown)} \end{aligned}$$

$$\begin{aligned} S &= \frac{1}{2}\ell^2\left(\sin\theta-\sin^3\theta\right) \\ \frac{dS}{d\theta} &= \frac{1}{2}\ell^2\left(\cos\theta-3\sin^2\theta\cos\theta\right) \\ &= \frac{1}{2}\ell^2\cos\theta\left(1-3\sin^2\theta\right) \end{aligned}$$

$$\frac{dS}{d\theta} = 0 \Rightarrow \frac{1}{2}\ell^2\cos\theta\left(1-3\sin^2\theta\right) = 0$$

$\cos\theta = 0$ (rejected) or $\sin\theta = \frac{1}{\sqrt{3}}$ or $\sin\theta = -\frac{1}{\sqrt{3}}$ (rejected)

since $0 < \theta < \frac{\pi}{2}$

$$\begin{aligned} \frac{dS}{d\theta} &= \frac{1}{2}\ell^2\cos\theta\left(1-3\sin^2\theta\right) \\ \frac{d^2S}{d\theta^2} &= \frac{1}{2}\ell^2\left\{(\cos\theta)(-6\sin\theta\cos\theta)+\left(1-3\sin^2\theta\right)(-\sin\theta)\right\} \\ &= -\frac{1}{2}\ell^2\sin\theta\left(6\cos^2\theta+1-3\sin^2\theta\right) \\ &= -\frac{1}{2}\ell^2\sin\theta\left(9\cos^2\theta-2\right) \quad \text{or} \quad -\frac{1}{2}\ell^2\sin\theta\left(7-9\sin^2\theta\right) \end{aligned}$$

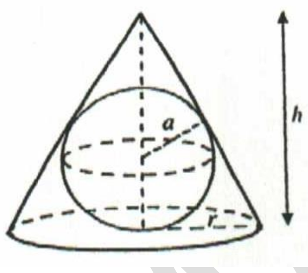
When $\sin\theta = \frac{1}{\sqrt{3}}$, $\cos^2\theta = \frac{2}{3}$, $\frac{d^2S}{d\theta^2} < 0$

$\therefore S$ is max when $\sin\theta = \frac{1}{\sqrt{3}}$

$$\max S = \frac{1}{2}\ell^2\left(\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{3}}\right) = \frac{1}{2}\ell^2\left(\frac{2}{3}\right)\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{9}\ell^2$$

Q3

Question



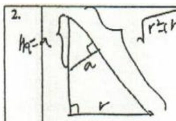
The diagram shows a sphere with fixed radius a which is inscribed in a cone with radius r and height h. Show that $r^2 = \frac{a^2h}{h-2a}$ and find the volume of cone, V in terms of a and h.

As h varies, find the value of h that gives the minimum V, leaving your answer in terms of a. (You do not need to verify that V is the minimum.)

[Volume of cone, $V = \frac{1}{3}\pi r^2 h$]

Solution

2.



$$\frac{r}{\sqrt{r^2+h^2}} = \frac{a}{h-a}$$

$(h-a)(r)=a\sqrt{r^2+h^2}$

$\Rightarrow (h-a)^2(r)^2=a^2(r^2+h^2)$

$r^2(h^2-2ah+a^2-a^2)=a^2h^2$

$r^2=\frac{a^2h^2}{h^2-2ah}=\frac{a^2h}{h-2a}$

$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{a^2h}{h-2a}\right)h$

$V=\frac{\pi a^2h^2}{3(h-2a)}$

$\frac{dV}{dh}=\frac{\pi a^2}{3}\cdot\frac{(h-2a)(2h)-h^2}{(h-2a)^2}$

$=\frac{\pi a^2}{3}\cdot\frac{h^2-4ah}{(h-2a)^2}$

When V is maximum, $\frac{dV}{dh}=0$

$h=0\,(rej)$, $h=4a$

Q4

Question

The curve C has parametric equations

$x=2+2\cos t$, $y=\tan t$, for $0\le t<\frac{\pi}{2}$

(i) Using differentiation, show that the curve C does not have any stationary point.

(ii) Sketch C, indicating clearly the equation of the asymptote and coordinates of the x-intercept. You should indicate clearly the feature of the curve near the x-intercept.

(iii) A point P on C has parameter $t=\frac{\pi}{4}$. Find the equation of the tangent at P, leaving your answer in exact form.

(iv) Find the Cartesian equation of the locus of the mid-point of $(2+2\cos t, \tan t)$ and $(-2,0)$ as t varies.

Solution

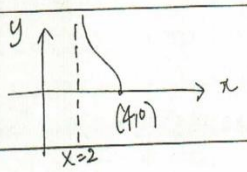
5(ai)

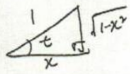
$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sec^2 t}{-2\sin t}=\frac{1}{-2\sin t\cos^2 t}$

Since $\frac{dy}{dx}<0$ for all $t\in\mathbb{R}$

$\therefore$ Curve C has no stationary point.

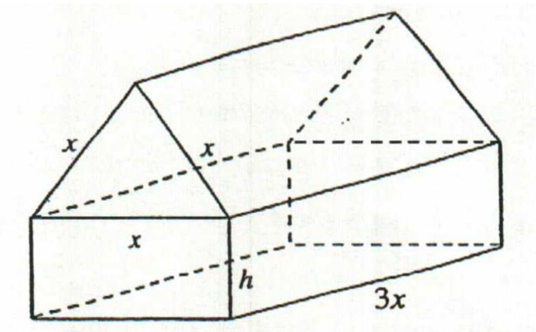
(aii)



(aiii)	At $t=\frac{\pi}{4}$, $x=2+\sqrt{2}$, $y=1$, $\frac{dy}{dx}=-\sqrt{2}$ $y-1=-\sqrt{2}(x-2-\sqrt{2})$ $y=-\sqrt{2}x+2\sqrt{2}+3$
	$M\ (\cos t, \frac{1}{2}\tan t)$ Let $x=\cos t$, $y=\frac{1}{2}\tan t$.  $y=\frac{1}{2}\left(\frac{\sqrt{1-x^2}}{x}\right)$ OR using trigo identity, $1+\tan^2 t=\sec^2 t$ $1+4y^2=\frac{1}{x^2}$ $y=\frac{1}{2}\left(\frac{\sqrt{1-x^2}}{x}\right)$, since $y>0$ for $0\le t<\frac{\pi}{2}$.

Q5

Question



An event company builds a tent (as shown in the diagram) which has a uniform cross-sectional area consisting of an equilateral triangle of sides x metres and a rectangle of width x metres and height h metres. The length of the tent is $3x$ metres.

(i) It is given that the tent has a fixed volume of k cubic metres and is fully covered (except the base) with canvas assumed to be of negligible thickness. Show that the area of the canvas used, A, in square metres, is given by $A = 6x^2 - \frac{3\sqrt{3}}{2}x^2 + \frac{8k}{3x}$
Use differentiation to find, in terms of k, the value of x which gives a stationary value of A. Determine if A is a minimum or maximum for this value of x.

(ii) It is given instead that the tent has a volume of 360 cubic metres and the area of canvas used is 300 square metres. Find the value of x and the value of h.

Solution

Qn	Suggested Solution
2(i)	$k=\left(\frac{1}{2}x^2\sin\frac{\pi}{3}+hx\right)(3x)$ $=\frac{3\sqrt{3}}{4}x^3+3hx^2$ $\therefore h=\frac{1}{3x^2}\left(k-\frac{3\sqrt{3}}{4}x^3\right)=\frac{k}{3x^2}-\frac{\sqrt{3}}{4}x$ $A=2\left(\frac{1}{2}x^2\sin\frac{\pi}{3}\right)+2(3x^2)+2(hx)+2(3hx)$ $=\frac{\sqrt{3}}{2}x^2+6x^2+8hx$ $=\frac{\sqrt{3}}{2}x^2+6x^2+8x\left(\frac{k}{3x^2}-\frac{\sqrt{3}}{4}x\right)$ $=\frac{\sqrt{3}}{2}x^2+6x^2+\frac{8k}{3x}-2\sqrt{3}x^2$ $=6x^2-\frac{3\sqrt{3}}{2}x^2+\frac{8k}{3x}$ (shown)

2

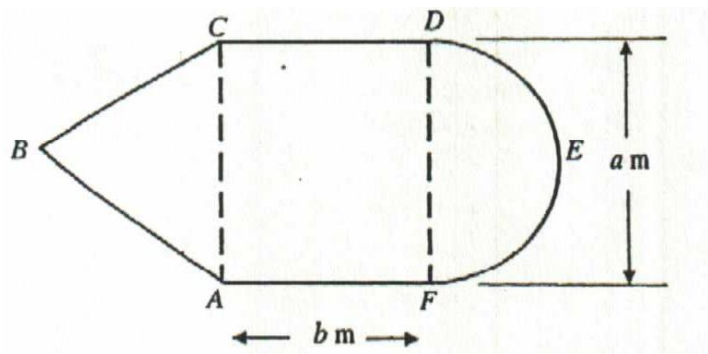
	$\therefore \frac{dA}{dx}=12x-3x\sqrt{3}-\frac{8k}{3x^2}$ For stationary values, $\frac{dA}{dx}=12x-3x\sqrt{3}-\frac{8k}{3x^2}=0$ $12x-3x\sqrt{3}-\frac{8k}{3x^2}=0$ $9x^3\left(4-\sqrt{3}\right)=8k$ $x^3=\frac{8k}{9\left(4-\sqrt{3}\right)}$ $\therefore x=\left(\frac{8k}{9\left(4-\sqrt{3}\right)}\right)^{\frac{1}{3}}$ $\frac{d^2A}{dx^2}=12-3\sqrt{3}+\frac{16k}{3x^3}$ Since $x^3>0, k>0, 12-3\sqrt{3}>0$ Alternative $\frac{d^2A}{dx^2}=12-3\sqrt{3}+\frac{16k}{3x^3}$ $=12-3\sqrt{3}+\frac{16k}{3}\left(\frac{9\left(4-\sqrt{3}\right)}{8k}\right)$ $=12-3\sqrt{3}+6\left(4-\sqrt{3}\right)$ $=36-9\sqrt{3}>0$. $\therefore$ area A is a minimum.
2(ii)	Using $k=360$ and $A=300$, $300=6x^2-\frac{3\sqrt{3}}{2}x^2+\frac{8(360)}{3x}$ $1800x=36x^3-9\sqrt{3}x^3+5760$ $x^3-88.185x+282.19=0$ From GC, since $x>0$, $x=3.8442$ or $x=6.8587$ When $x=3.8442$, $h=\frac{360}{3(3.8442)^2}-\frac{\sqrt{3}}{4}(3.8442)=6.46$.

	When $x=6.8587$, $h=\frac{360}{3(6.8587)^2}-\frac{\sqrt{3}}{4}(6.8587)=-0.419$ (rej. $\because h>0$) $\therefore x=3.84,\ h=6.46$.

Q6

Question

The diagram shows a field consisting of an equilateral triangle ABC, a rectangle $ACDF$ and a semi-circle DEF.



Suppose $DF = a$ m, $AF = b$ m, and the area of the field is fixed at 400 m^2.

Find, using differentiation, the values of a and b which give a field of minimum perimeter, giving your answers correct to 2 decimal places.

Solution

5. $\text{Area} = \frac{1}{2}\pi\left(\frac{a}{2}\right)^2 + ab + \frac{1}{2}a^2\sin 60^\circ = \frac{\pi a^2}{8} + ab + \frac{\sqrt{3}}{4}a^2$

Thus $\frac{\pi a^2}{8} + ab + \frac{\sqrt{3}}{4}a^2 = 400$

$\Rightarrow b = \dfrac{400 - \frac{\pi a^2}{8} - \frac{\sqrt{3}}{4}a^2}{a} = \frac{400}{a} - \frac{\pi}{8}a - \frac{\sqrt{3}}{4}a$ --- (1)

Perimeter, $P = 2a + 2b + \frac{\pi a}{2}$

Sub (1) $\quad = 2a + 2\left(\frac{400}{a} - \frac{\pi}{8}a - \frac{\sqrt{3}}{4}a\right) + \frac{\pi a}{2}$

$= \left(2 + \frac{\pi}{4} - \frac{\sqrt{3}}{2}\right)a + \frac{800}{a}$

$\frac{dP}{da} = 2 + \frac{\pi}{4} - \frac{\sqrt{3}}{2} - \frac{800}{a^2}$

When $\frac{dP}{da} = 0$, $\Rightarrow 2 + \frac{\pi}{4} - \frac{\sqrt{3}}{2} - \frac{800}{a^2} = 0$

$\Rightarrow a^2 = \dfrac{800}{2 + \frac{\pi}{4} - \frac{\sqrt{3}}{2}}$

Since $a > 0$, therefore $a = \sqrt{\dfrac{800}{2 + \frac{\pi}{4} - \frac{\sqrt{3}}{2}}}$

$= 20.416 = 20.42$ m (correct to 2 d.p)

$b = \frac{400}{a} - \frac{\pi}{8}a - \frac{\sqrt{3}}{4}a = 2.735 = 2.74$ m (correct to 2 d.p)

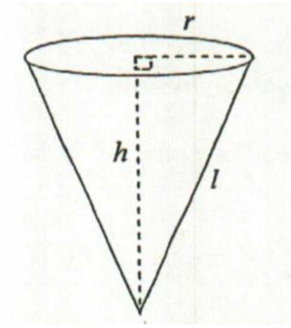
Check for minimum using 2$^{\text{nd}}$ derivative test:

$\frac{d^2P}{da^2} = \frac{1600}{a^3} > 0 \therefore P$ is minimum.

Q7

Question

[It is given that a cone of radius r, height h and slant length l has volume $\frac{1}{3}\pi r^2 h$ and curved surface area $\pi r l$.]



An ice cream cone wafer (as shown in the diagram above) of negligible thickness is to have a fixed external surface area of $k\pi$ cm^2. Show that the volume V of the cone is given by

$$V = \frac{\pi r\sqrt{k^2 - r^4}}{3}$$

Use differentiation to find the radius r cm of the cone in terms of k that will give a minimum internal volume of the cone (you need not prove the minimum value of V).

Solution

5

$$\pi r l = k\pi$$

$$l = \frac{k}{r}$$

$$l^2 = r^2 + h^2$$

$$h = \sqrt{\frac{k^2}{r^2} - r^2} = \frac{\sqrt{k^2 - r^4}}{r}$$

$$V = \frac{1}{3}\pi r^2\left(\frac{\sqrt{k^2 - r^4}}{r}\right)$$

$$V = \frac{\pi r\sqrt{k^2 - r^4}}{3}$$

$$\frac{dV}{dr} = \frac{\pi\sqrt{k^2 - r^4}}{3} - \frac{2\pi r^4}{3\sqrt{k^2 - r^4}}$$

At stationary point, $\frac{dV}{dr} = 0$

$$\frac{\pi\sqrt{k^2 - r^4}}{3} = \frac{2\pi r^4}{3\sqrt{k^2 - r^4}}$$

$$k^2 - r^4 = 2r^4$$

$$3r^4 = k^2$$

$$r = \sqrt[4]{\frac{k^2}{3}}$$

Q8

Question

The parametric equations of a curve C are

$x = \cos^3\theta, y = 2\sin^3\theta$ where $-\frac{\pi}{2} \leq \theta \leq 0$

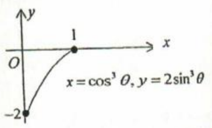
(i) Sketch the graph of C

(ii) Find $\frac{dy}{dx}$ in terms of θ

(iii) The tangent to the curve C at the point $P(\cos^3 t, 2\sin^3 t)$ intersects the x-axis and the y-axis at the points U and V respectively. Show that the coordinates of U are $(a\cos t, 0)$, where a is a constant to be found. Find the coordinates of V.

(iv) Find a Cartesian equation of the locus of the mid-point of UV as t varies.

Solution

Qn	Solution
10	**Tangent and Normal – Parametric Equations**
(i)	
(ii)	$\frac{dy}{dx}=\frac{6\sin^2\theta\cos\theta}{-3\cos^2\theta\sin\theta}=-2\tan\theta$
(iii)	Equation of tangent at $P(\cos^3 t, 2\sin^3 t)$ is $y-2\sin^3 t=-2\tan t(x-\cos^3 t)$ At $x=0$, $y=-2\tan t(-\cos^3 t)+2\sin^3 t$ $=2\sin t\cos^2 t+2\sin^3 t$ $=2\sin t(1-\sin^2 t)+2\sin^3 t$ $=2\sin t$ At $y=0$, $-2\sin^3 t=-2\tan t(x-\cos^3 t)$ $x(2\tan t)=2\tan t\cos^3 t+2\sin^3 t$ $x(2\tan t)=2\sin t$ $x=\frac{2\sin t}{2\tan t}=\cos t$ Therefore $a=1$ Coordinates of U and V are $(\cos t, 0)$ and $(0, 2\sin t)$
(iv)	Midpoint of UV is $\left(\frac{\cos t}{2}, \frac{2\sin t}{2}\right)=\left(\frac{\cos t}{2}, \sin t\right)$ $x=\frac{\cos t}{2},\ y=\sin t$ $2x=\cos t,\ y=\sin t$ $\cos^2 t+\sin^2 t=1$ $4x^2+y^2=1$ Therefore, cartesian equation of the locus of the mid-point of UV as t varies is $4x^2+y^2=1$ where $-1\le y\le 0,\ 0\le x\le 0.5$.

Q9

Question

Benjamin wants to make an open fish tank in the shape of a cuboid with a square base. The base of the fish tank is to be made from an opaque material which costs \$5 per square centimetre. The sides of the fish tank are to be made from a transparent material which costs \$2 per square centimetre. All the materials used are of negligible thickness.

If the fish tank has a capacity of 10000 cm^3, find the minimum cost of the fish tank.

Solution

2	$10000 = x^2 h$ $h = \frac{10000}{x^2}$ Let C be the total cost in dollars. $C = 5x^2 + 2(4xh)$ $C = 5x^2 + \frac{80000}{x}$ $\frac{dC}{dx} = 10x - \frac{80000}{x^2}$ Let $\frac{dC}{dx} = 0$, $x = 20$ $\frac{d^2C}{dx^2} = 10 + \frac{160000}{x^3}$ $\left.\frac{d^2C}{dx^2}\right\vert_{x=20} = 30 > 0$ Hence, C is minimum when $x = 20$. $C_{min} = 6000$

Q10

Question

A curve C has parametric equations

$x = \ln t$, $y = 2 - \frac{1}{2t}$

(i) Show that the equation of the tangent to the curve C at the point with parameter p is $y = \frac{x}{2p} - \frac{\ln p}{2p} - \frac{1}{2p} + 2$

(ii) Let A be the point on the curve C with parameter 1. The tangent and normal at A intersect the x-axis at the points T and N respectively. Find the coordinates of the points T and N and the area of the triangle ANT.

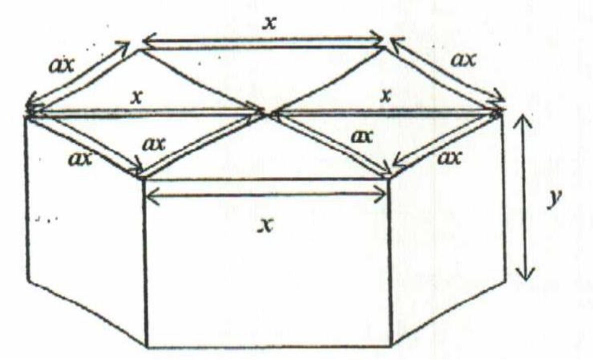
Solution

(iii)	
4 (i)	$\frac{dy}{dx} = \frac{1}{2t}$ Equation of tangent at the point with parameter p: $y - \left(2 - \frac{1}{2p}\right) = \frac{1}{2p}(x - \ln p)$ $y = \frac{x}{2p} - \frac{\ln p}{2p} - \frac{1}{2p} + 2$
4 (ii)	Equation of tangent at the A (i.e. $p = 1$): $y = \frac{x}{2} + \frac{3}{2}$ When $y = 0$, $x = -3$ $T(-3,0)$ Equation of normal at A: $y = -2x + \frac{3}{2}$ When $y = 0$, $x = \frac{3}{4}$ $N\left(\frac{3}{4}, 0\right)$ Area of $\Delta ANT = \frac{45}{16}$

Q11

Question

A company requires a box made of cardboard of negligible thickness to hold 300 cm^3 of powder when full. The top and the base of the box are made up of six identical isosceles triangles. The two identical sides of the isosceles triangle are of length ax cm, where a is a constant and $> \frac{1}{2}$, and the remaining side is of length x cm. The height of the box is y cm (see diagram).



(i) Use differentiation to find, in terms of a, the value of x which gives a minimum surface area of the box.

(ii) Show that, in this case, $\frac{y}{x} = 3\sqrt{\frac{2a-1}{2a+1}}$. Hence find the range of $\frac{y}{x}$.

Solution

5	Height of an isosceles $\sqcup = \sqrt{(ax)^2 - \left(\frac{x}{2}\right)^2} = x\sqrt{a^2 - \frac{1}{4}}$ Area of one $\sqcup = \frac{1}{2}x\left(x\sqrt{a^2 - \frac{1}{4}}\right) = \frac{x^2}{2}\sqrt{a^2 - \frac{1}{4}}$ Area of base $= 6 \cdot \frac{x^2}{2}\sqrt{a^2 - \frac{1}{4}} = 3x^2\sqrt{a^2 - \frac{1}{4}}$ $300 = \left(3x^2\sqrt{a^2 - \frac{1}{4}}\right)y = \left(3\sqrt{a^2 - \frac{1}{4}}\right)x^2 y \cdots\cdots (1)$ Surface Area, $S = \left(3x^2\sqrt{a^2 - \frac{1}{4}}\right)2 + 2xy + 4axy$ From (1), $y = \frac{100}{\left(\sqrt{a^2 - \frac{1}{4}}\right)x^2}$ Therefore

$$S=\left(3x^2\sqrt{a^2-\frac{1}{4}}\right)2+2x\frac{100}{\left(\sqrt{a^2-\frac{1}{4}}\right)x^2}+4ax\frac{100}{\left(\sqrt{a^2-\frac{1}{4}}\right)x^2}$$

$$=\left(6\sqrt{a^2-\frac{1}{4}}\right)x^2+2\frac{100}{\left(\sqrt{a^2-\frac{1}{4}}\right)x}+4a\frac{100}{\left(\sqrt{a^2-\frac{1}{4}}\right)x}$$

$$=\left(6\sqrt{a^2-\frac{1}{4}}\right)x^2+\left(\frac{200}{\sqrt{a^2-\frac{1}{4}}}+\frac{400a}{\sqrt{a^2-\frac{1}{4}}}\right)\frac{1}{x}$$

$$\frac{dS}{dx}=2\left(6\sqrt{a^2-\frac{1}{4}}\right)x-\left(\frac{200}{\sqrt{a^2-\frac{1}{4}}}+\frac{400a}{\sqrt{a^2-\frac{1}{4}}}\right)\frac{1}{x^2}$$

Let $\frac{dS}{dx}=0$

$$2\left(6\sqrt{a^2-\frac{1}{4}}\right)x-\left(\frac{200}{\sqrt{a^2-\frac{1}{4}}}+\frac{400a}{\sqrt{a^2-\frac{1}{4}}}\right)\frac{1}{x^2}=0$$

$$2\left(6\sqrt{a^2-\frac{1}{4}}\right)x=\left(\frac{200}{\sqrt{a^2-\frac{1}{4}}}+\frac{400a}{\sqrt{a^2-\frac{1}{4}}}\right)\frac{1}{x^2}$$

$$x^3=\frac{200}{12\left(a^2-\frac{1}{4}\right)}\cdot(1+2a)$$

$$x^3=\frac{200}{12\left(a+\frac{1}{2}\right)\left(a-\frac{1}{2}\right)}\cdot(1+2a)$$

$$x^3=\frac{200}{3(2a+1)(2a-1)}\cdot(1+2a)=\frac{200}{3(2a-1)}$$

$$x=\left(\frac{200}{3(2a-1)}\right)^{\frac{1}{3}}$$

$$\frac{d^2S}{dx^2}=2\left(6\sqrt{a^2-\frac{1}{4}}\right)+2\left(\frac{200}{\sqrt{a^2-\frac{1}{4}}}+\frac{400a}{\sqrt{a^2-\frac{1}{4}}}\right)\frac{1}{x^3}>0$$

as $x>0\Rightarrow x^3>0$ and $a>\frac{1}{2}\Rightarrow a^2-\frac{1}{4}>0$

Recall that $y=\frac{100}{\left(\sqrt{a^2-\frac{1}{4}}\right)x^2}$

therefore $\frac{y}{x}=\frac{100}{\left(\sqrt{a^2-\frac{1}{4}}\right)x^3}=\frac{100}{\left(\sqrt{a^2-\frac{1}{4}}\right)\left(\frac{200}{3(2a-1)}\right)}$

$$=\frac{3}{\left(\sqrt{a^2-\frac{1}{4}}\right)\left(\frac{2}{(2a-1)}\right)}=\frac{3(2a-1)}{\sqrt{4a^2-1}}=\frac{3\sqrt{2a-1}}{\sqrt{2a+1}}=3\sqrt{\frac{2a-1}{2a+1}}$$

Method 1(Graphical):

Since $a>\frac{1}{2}$, therefore $0<\frac{2a-1}{2a+1}<1$ or $0<\sqrt{\frac{2a-1}{2a+1}}<1$, [Draw graph to show],

Therefore $0<3\sqrt{\frac{2a-1}{2a+1}}<3$

Method 2(Algebraic)

$$\frac{3\sqrt{2a-1}}{\sqrt{2a+1}}=3\sqrt{\frac{2a-1}{2a+1}}=3\sqrt{1-\frac{2}{2a+1}}$$

Since $a>\frac{1}{2}$, $\Rightarrow 2a+1>2>0$ $\quad\Rightarrow 0<\frac{1}{2a+1}<\frac{1}{2}$

$\Rightarrow 0>-\frac{2}{2a+1}>-1$

$\Rightarrow 1>1-\frac{2}{2a+1}>0$

$\Rightarrow 0<\sqrt{1-\frac{2}{2a+1}}<1$ $\Rightarrow 0<3\sqrt{1-\frac{2}{2a+1}}<3$

Q12

Question

The curve C has parametric equations $=\frac{t}{t^2-a}$, $y=te^{-t}$, where a is a positive real constant and $-\sqrt{a}<t\le 0$.

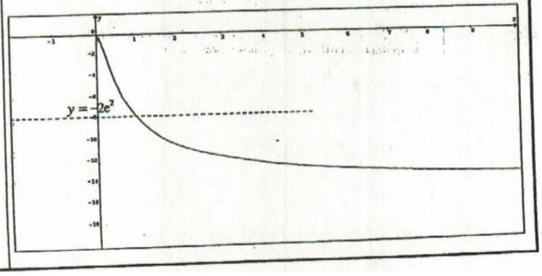
(i) The tangent to the curve C at $t=0$ is perpendicular to the line $4y-x=0$. Show that $a=4$.

Using the value of a in part (i),

(ii) What can you say about the gradient of the curve C as $\rightarrow -2$?

(iii) Sketch the curve C, including any points of intersection with the axes and the equation(s) of any asymptotes.

Solution

3 (i)	$x=\frac{t}{t^2-a}$, $y=te^{-t}$ $\frac{dx}{dt}=\frac{t^2-a-t(2t)}{(t^2-a)^2}=\frac{-t^2-a}{(t^2-a)^2}$ $\frac{dy}{dt}=-te^{-t}+e^{-t}=e^{-t}(1-t)$ $\frac{dy}{dx}=\frac{e^{-t}(1-t)(t^2-a)^2}{-t^2-a}$ At $t=0$, $\frac{dy}{dx}=\frac{(-a)^2}{-a}=-a$. Since the tangent to the curve C at $t=0$ is perpendicular to the line $4y-x=0$, $\frac{1}{4}(-a)=-1\Rightarrow a=4$ (Shown)
3 (ii)	For $a=4$, $\frac{dy}{dx}=\frac{e^{-t}(1-t)(t^2-4)^2}{-t^2-4}$. As $t\rightarrow -2$, $\frac{dy}{dx}\rightarrow\frac{e^2(3)(4-4)^2}{-4-4}=0$. The gradient of the curve approaches 0 as $t\rightarrow -2$.
3 (iii)	Observe that as $t\rightarrow -2$, $x=\frac{t}{t^2-4}\rightarrow\infty$, $y=te^{-t}\rightarrow -2e^2$. Thus, $y=-2e^2$ is a horizontal asymptote. 

Q13

Question

A function f is given by $f(x) = \frac{x^2 - ax + b}{x}$ for $\in \mathbf{R}$, $x \neq 0$ where a and b are positive real constants.

Find $f'(x)$ and hence sketch the graph of $= f'(x)$, stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes.

Solution

Qn. [Marks]	Solution
1 [4]	$f(x) = \frac{x^2 - ax + b}{x} = x - a + \frac{b}{x}$ $f'(x) = 1 - \frac{b}{x^2} = 0 \Leftrightarrow x = \pm\sqrt{b}$ x-intercepts $(\sqrt{b}, 0)$ *and* $(-\sqrt{b}, 0)$